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NOTES

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Monotone Multiplicative Functions

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The purpose of this note is to present a simple and intuitive proof of the following classical result **[2]** about multiplicative functions defined on the natural numbers $\mathbb{N}$. Recall first that a function $f\colon \mathbb{N} \to \mathbb{R}^+$ is **multiplicative** if $f(mn) = f(m)f(n)$ whenever m and n are relatively prime, i.e., if the greatest common divisor, (m, n), is 1.

THEOREM. *If $f\colon \mathbb{N} \to \mathbb{R}^+$ is a monotone multiplicative function, then there exists some r such that $f(n) = n^r$ for all n.*

Reference **[3]** contains a discussion of the problem and of some earlier proofs. Our proof is more conceptual than those listed in the bibliography and is based on one simple observation: a multiplicative function $f\colon \mathbb{N} \to \mathbb{R}^+$ can be extended uniquely to a "multiplicative function" on $\mathbb{Q}^+$. If f is increasing on $\mathbb{N}$, then it is also increasing on $\mathbb{Q}^+$ and can be extended to an increasing function on $\mathbb{R}^+$, which is almost totally multiplicative. It is then easy to prove that f is continuous and totally multiplicative, and it is well known that a continuous endomorphism of $\mathbb{R}^+$ is necessarily raising to a power. For the proof, we need the following well-known fact from real analysis: An increasing function from the reals to themselves has a countable set of discontinuities. Except for this, the paper is self-contained.

First, whenever we write a rational number $p/q \in \mathbb{Q}^+$, it will be assumed that p and q are positive integers such that $(p, q) = 1$. Two rational numbers $\alpha = p/q$ and $\beta = r/s$ are **relatively prime**, written $(\alpha, \beta) = 1$, if p, q, r, and s are relatively prime in pairs. A function $f\colon \mathbb{Q}^+ \to \mathbb{R}^+$ is called **multiplicative** if $f(\alpha\beta) = f(\alpha)f(\beta)$ whenever $(\alpha, \beta) = 1$.

LEMMA. *Every given positive rational number p/q (and hence every given positive real number) can be approximated arbitrarily closely from above and from below by rational numbers that are relatively prime to any given finite set of positive rational numbers.*

Proof. Let S be the given finite set, and let m be the product of pq and all the numerators and denominators in S. Let n be an arbitrary positive integer. Then the two rational numbers $r_\varepsilon = (pnm + \varepsilon)/(qnm - \varepsilon)$, where $\varepsilon = \pm 1$, are relatively prime to p/q and to each rational number in S, and they satisfy the inequalities $r_{-1} < p/q < r_{+1}$. Moreover each differs from p/q by less than $(p + q)/(qnm - 1)$, which can be made arbitrarily small by choosing n large enough.

Let $f\colon \mathbb{N} \to \mathbb{R}^+$ be an increasing multiplicative function. For $\alpha = p/q \in \mathbb{Q}^+$, define $f(\alpha) = f(p)/f(q)$. It is immediate that $f\colon \mathbb{Q}^+ \to \mathbb{R}^+$ is multiplicative. We now show that it is also increasing: If $\alpha = p/q < \beta = r/s$ and if $(\alpha, \beta) = 1$, then $sp < rq$, so that $f(s)f(p) = f(sp) \le f(rq) = f(r)f(q)$, whence $f(\alpha) \le f(\beta)$. If α

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and β are not relatively prime, by the lemma we can find γ such that $\alpha < \gamma < \beta$, and $(\alpha, \gamma) = (\gamma, \beta) = 1$. Thus $f(\alpha) \leq f(\gamma) \leq f(\beta)$. So f is increasing and multiplicative. We can extend it to $\mathbb{R}^+$ as follows: for η irrational, let $f(\eta) = \lim_{\alpha \to \eta^-} f(\alpha)$, the limit taken from $\alpha \in \mathbb{Q}^+$. Note that f is increasing and is continuous from the left at irrationals.

Let $\eta, \theta \in \mathbb{R}^+$ with θ irrational. Using the lemma, we can find α and β relatively prime rational numbers less than but arbitrarily close to η and θ respectively, except that if η is rational we take $\alpha = \eta$. Then $f(\alpha)f(\beta) = f(\alpha\beta)$. Passing to limits we get $f(\eta)f(\theta) = f(\eta\theta)$. Thus f is multiplicative except possibly on the product of rational numbers not relatively prime.

Now f is increasing on $\mathbb{R}^+$, so it has a countable set of discontinuities. In particular, if δ is any real number, then we can find an irrational number ξ such that f is continuous at $\delta\xi$. Since ξ is irrational, $f(\alpha) = f(\alpha\xi)/f(\xi)$ for any real number α. Thus $\lim_{\alpha \to \delta} f(\alpha) = \lim_{\alpha \to \delta} f(\alpha\xi)/f(\xi) = f(\delta\xi)/f(\xi) = f(\delta)$. Thus f is continuous everywhere. If α and β are rational, then approach β by irrational numbers θ. Since f is continuous, $f(\alpha\beta) = \lim_{\theta \to \beta} f(\alpha\theta) = \lim_{\theta \to \beta} f(\alpha)f(\theta) = f(\alpha)f(\beta)$, so f is totally multiplicative. Thus $\log \circ f \circ \exp\colon \mathbb{R} \to \mathbb{R}$ is a continuous homomorphism and so by elementary linear algebra is multiplication by some constant r. So $f(x) = x^r$ and the theorem is proved for the case of f increasing. For f decreasing $1/f$ is increasing and we obtain the same result.

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Powers of a Prime Dividing Binomial Coefficients

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The binomial coefficients

$$C(n, k) = \frac{n!}{k!(n-k)!}$$

form an enduring source of interesting problems and results. In a recent Monthly article **[1]**, Goetgheluck explained how patterns calculated with the aid of a microcomputer led him to rediscover a result of Kummer [3] giving the exact power of a prime p dividing $C(n, k)$, enabling him to give a fast way of computing $C(n, k)$ for large values of n. In this note, I give a result about the set of powers of p exactly dividing the $C(n, k)$, for $0 \leqslant k \leqslant n$, suggested by thinking about a