A NEW PROOF OF ERDŐS'S THEOREM ON MONOTONE MULTIPLICATIVE FUNCTIONS

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1. Introduction. An arithmetical function f, not identically zero, is said to be *multiplicative* if

$$f(mn) = f(m)f(n) \quad \text{whenever} \quad (m,n) = 1,$$

and *completely multiplicative* if

$$f(mn) = f(m)f(n) \quad \text{for all } m \text{ and } n.$$

The following remarkable theorem concerning increasing multiplicative functions is due to Erdős [**2**].

THEOREM. *If f is increasing and multiplicative, then there is a constant α such that $f(n) = n^\alpha$ for all $n \geqslant 1$.*

Erdős's original proof is rather complicated, and simpler proofs have been given by Moser and Lambek [**3**], Besicovitch [**1**], and Schoenberg [**4**]. All of these proofs are either lengthy or not well motivated. This paper shows that the result is fairly easy to prove for completely multiplicative functions; the difficulty lies in showing that every increasing multiplicative function is completely multiplicative. Consequently, we split Erdős's theorem into two parts as follows:

THEOREM A. *If f is increasing and completely multiplicative, then there is a constant α such that $f(n) = n^\alpha$ for all $n \geqslant 1$.*

THEOREM B. *Every increasing multiplicative function is completely multiplicative.*

2. Proof of Theorem A. Let f be increasing and completely multiplicative. We prove Theorem A by contradiction. Assume there is no α such that $f(n) = n^\alpha$ for all $n \geqslant 1$. Then $\log f(n)/\log n$ is not constant, so there exist distinct integers $m > 1$, $n > 1$, such that

$$\frac{\log f(m)}{\log m} > \frac{\log f(n)}{\log n}.$$

Taking x to be the larger and y the smaller of these two ratios, we have

$$f(m) = m^x \quad \text{and} \quad f(n) = n^y,$$

with $x > y$.

Because $y/x < 1$ there exist integers $A \geqslant 1$ and $B \geqslant 1$ such that

$$\frac{y}{x} B \frac{\log n}{\log m} < A \leqslant B \frac{\log n}{\log m}.$$

In fact,

$$B = \left[\frac{x \log m}{(x-y)\log n} \right] + 1 \quad \text{and} \quad A = \left[B \frac{\log n}{\log m} \right]$$

satisfy the above inequalities. But then we have both

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$$A \log m \leqslant B \log n \quad \text{and} \quad Ax \log m > By \log n,$$

or

$$m^A \leqslant n^B \quad \text{and} \quad m^{Ax} > n^{By}.$$

However, since f is completely multiplicative,

$$f(m^A) = f(m)^A = m^{Ax} \quad \text{and} \quad f(n^B) = f(n)^B = n^{By},$$

so that $m^A \leqslant n^B$ while $f(m^A) > f(n^B)$, contradicting the fact that f is increasing. This proves Theorem A.

3. Proof of Theorem B. The proof of Theorem B is based on the following lemma.

LEMMA 1. *Given an increasing multiplicative function f and a prime p, let*

$$L = \inf_{x \not\equiv 0 \bmod p} \frac{f(x+p)}{f(x)}.$$

Then $L = 1$.

We use Lemma 1 to deduce Theorem B, and then we prove Lemma 1 in the next section.

Assume f is increasing and multiplicative. To show that f is completely multiplicative it suffices to show that for every prime p and every integer $n > 1$ we have

$$f(p^n) = f(p)^n.$$

Fix a prime p and let $y_n = f(p^n)$. We will show that $y_n = y_1^n$ or, equivalently, that

$$\frac{y_{n+1}}{y_n} = y_1 \tag{1}$$

for all n.

Consider any integer x not divisible by p. Then for any n we have

$$(px-1)p^n < xp^{n+1} < (px+1)p^n.$$

Now each of $px-1$, x, and $px+1$ is prime to p so

$$f(px-1)y_n \leqslant f(x)y_{n+1} \leqslant f(px+1)y_n,$$

because f is increasing and multiplicative. Therefore,

$$\frac{y_{n+1}}{y_n} \leqslant \inf \frac{f(px+1)}{f(x)} \leqslant \inf \frac{f(px+p^2)}{f(x)} = f(p) \inf \frac{f(x+p)}{f(x)} = y_1 L,$$

where the inf is taken over all $x \not\equiv 0 \bmod p$. Similarly, we find

$$\frac{y_{n+1}}{y_n} \geqslant y_1 U,$$

where

$$U = \sup_{\substack{x \not\equiv 0 \bmod p \\ x > p}} \frac{f(x-p)}{f(x)} = \sup_{x \not\equiv 0 \bmod p} \frac{f(x)}{f(x+p)} = \frac{1}{L}.$$

Thus we have

$$Uy_1 \leqslant \frac{y_{n+1}}{y_n} \leqslant Ly_1.$$

But $L = 1$ by Lemma 1, so $U = 1$ also, and this last inequality implies (1), which, in turn, proves Theorem B.

4. Proof of Lemma 1. Assume f is increasing and multiplicative. From the definition of L, if an integer x is not divisible by p we have $f(x+p) \geqslant Lf(x)$, and hence

$$f(x+kp) \geqslant L^k f(x)$$

for every integer $k \geqslant 0$.

Now, given any $k \geqslant 0$, we can find an integer $x > kp$ which is prime to both p and 2. Then $2x > x + kp$, and

$$f(2)f(x) = f(2x) \geqslant f(x+kp) \geqslant L^k f(x),$$

so that $L^k \leqslant f(2)$. Since k was arbitrary, we must have $L \leqslant 1$. But since f is increasing, we also have $L \geqslant 1$. These two inequalities show that $L = 1$ and prove Lemma 1.

5. Comments. Erdős's theorem implies a corresponding result for decreasing multiplicative functions with one restriction. It is easy to show that if f is decreasing and multiplicative, then either f is always positive, or else $f(1) = 1$ and $f(n) = 0$ when $n > 2$, with $0 \leqslant f(2) \leqslant 1$. Now, if f is positive and decreasing, then $1/f$ is increasing, so Erdős's theorem implies $1/f(n) = n^\alpha$ for some α, hence $f(n) = n^{-\alpha}$. Thus, if f is monotone and multiplicative and if $f(3) \neq 0$, then there exists a constant α such that $f(n) = n^\alpha$ for $n \geqslant 1$.

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GRAPHICAL CONSTRUCTIONS OF MEANS

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1. Introduction. In chapter three of *Inequalities* by Hardy, Littlewood, and Pólya, it is shown that every continuous and strictly monotonic function defined on an interval can be used to assign a mean value to each set of n numbers in the interval. The mean value assigned is only a mean in a general sense: the mean is not less than the smallest of the numbers nor greater than the largest. Moreover, properties that the familiar means possess need not hold for the mean defined by an arbitrary function. In this paper we show that the graphical technique for determining a mean of two variables given by Moskovitz in [4] can be generalized to determining a mean of n variables (in the sense of Hardy, Littlewood, and Pólya) through a graphical technique in R^n. This provides some especially interesting geometric representations of many well-known means in R^3 and more generally in R^n.

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