Universal Definitions of the Roman Factorial: Expansion to Multifactorials and the Omnifactorial

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August 21, 2024

Abstract

This paper expands on our previous research about the Roman factorial by investigating factorials of higher orders, starting with the double factorial. We explore an extension of the double factorial to negative integers, resulting in both recursive and non-recursive piece-wise definitions. We then unify them into universal representations using a set of Boolean-like functions called *foundational functions* through the *generalization process*. Our methodology is further applied to the triple factorial and factorials of higher orders, aiming to create a unified framework for understanding several variations of the factorial.

1 Preface

1.1 Introduction

This paper is a continuation of the previous work titled *Universal Definitions of the Roman Factorial: Introduction to Foundational Functions and the Generalization Process* [1]. It is second of a five-part series exploring the factorial function and its various extensions, as briefly introduced in Part 1.

1.2 Useful information

For those reading this document online, please note that all references to equations, figures, tables, and sections are hyperlinked. This feature enables instant navigation to the cited content without the need to memorize anything. The phrase "Part 1" consistently links to the previous paper, wherever it appears in this document.

Below is a short list of terms and concepts helpful for understanding this paper:

- **Factorial**: A mathematical function applied to natural numbers, defined as the product of all positive integers up to a given number.
- **Double factorial**: A variant of the factorial, multiplying every second integer from n down to 1 or 2, depending on whether n is odd or even.
- **Multifactorial**: A generalization of the double factorial, where the product is taken over every m-th positive integer up to n.
- **Roman factorial**: An extension of the factorial to negative integers.
- **Roman factorial of order m**: A further generalization of the Roman factorial to higher orders.
- **Omnifactorial**: An extension of the multifactorial that applies to negative integers as well.
- **Piece-wise definition**: A method of defining a function by partitioning its domain into distinct intervals, each governed by a specific formula or rule. Also: *closed form.*
- **Universal definition**: A single mathematical expression that uniformly applies to all input ranges. Also: *unified, global.*
- **Generalization process**: The method of integrating *Foundational Functions (F.F.)* into the piece-wise definitions of a function to consolidate them into a single expression.

The abbreviations used throughout this paper are outlined below:

- *F.F.*: *Foundational functions*, a series of simple Boolean-like functions that are built upon each other and usually have binary outputs (0 or 1).
- *Eq.*: An equation or a relationship.
- *Tbl.*: A mathematical table that is useful for presenting information in a clear and concise way.
- *Fig.*: A figure or a diagram, that here will often depict the behavior of a function in a domain close to 0.

The remainder of this preface introduces a new notation system for various factorial extensions and provides a summary of Part 1. If any unfamiliar terms or abbreviations arise, you may refer back to this section for clarification.

1.3 Notation

In this study, we examine several variations of the factorial. In Part 1, we introduced the Roman factorial [2], an extension of the traditional factorial to negative integers, and provided definitions both recursive and non-recursive.

Each of the five parts of this study further develops the concept of the factorial, necessitating precise notation to distinguish between these related concepts. In this section, we propose a framework that encompasses all relevant factorial variations discussed in this paper.

The notation system includes three basic expressions to describe a factorial, even though a single expression could technically suffice to characterize even the most generalized version.

For the Roman factorial, we suggest the following notation:

- As the commonly known factorial for $n \in \mathbb{Z}_0^+$, we define "$n!_{(1)}$" (termed **traditional factorial**). It is alternatively written as "$n!$" within this study and can simply be referred to as the **factorial** to preserve its original definition [3].
- As the expansion of the traditional factorial to $n \in \mathbb{Z}$ that is described by its original formulation by Steven Roman, we define "$\lfloor n \rceil !_1$" (termed **Roman factorial of order 1**, or **Roman factorial**). It is also written as "$\lfloor n \rceil !$" within this study, which is how it was initially presented.
- As the expansion of the traditional factorial to $n \in \mathbb{Z}$ that is described by other definitions (for instance, as a falling product), we define the expression "$n!_1$" (termed **factorial of order 1**).

Note that the last two expressions represent the same concept, although defined in different ways. For instance, $n!_1$ may be defined as a $\prod$-product, while $\lfloor n \rceil !_1$ refers to a specific formulation. Both yield identical results within the same domain.

Sections 2 to 4 address another factorial variation. For the double factorial, we propose:

- As the commonly known double factorial for $n \in \mathbb{Z}_0^+$, we define the expression "$n!_{(2)}$" (termed **double factorial**). It is alternatively written as "$n!!$" within this study for simplicity, which is also its original notation [4].
- As the expansion of the double factorial to $n \in \mathbb{Z}$, we define the expression "$n!_2$" (termed **factorial of order 2**). The difference between it and the double factorial is that it includes the domain of negative integers.
- As the factorial of order 2 described by a formulation similar to the Roman factorial, we define the expression "$\lfloor n \rceil !_2$" (termed **Roman factorial of order 2**).

Finally, we extend these notations to describe the multifactorial and its extension to negative integers. We propose the following definitions:

- As the commonly known multifactorial for $n \in \mathbb{Z}_0^+$, we define the expression "$n!_{(m)}$" (termed **multifactorial**).
- As the expansion of the multifactorial to $n \in \mathbb{Z}$, we define "$n!_m$" (termed **factorial of order m**). It is also called the **omnifactorial** within this study.
- As the factorial of order m described by a formulation similar to the Roman factorial, we define $\lfloor n \rceil !_m$" (termed **Roman factorial of order m**).

The table below summarizes these notations for clarity:

Symbol	Name	Set
$n!_{(1)}$, $n!$	traditional factorial	$\mathbb{Z}_0^+$
$n!_1$	factorial of order 1	$\mathbb{Z}$
$\lfloor n \rceil !_1$, $\lfloor n \rceil !$	Roman factorial	$\mathbb{Z}$
$n!_{(2)}$, $n!!$	double factorial	$\mathbb{Z}_0^+$
$n!_2$	factorial of order 2	$\mathbb{Z}$
$\lfloor n \rceil !_2$	Roman factorial of order 2	$\mathbb{Z}$
$n!_{(m)}$	multifactorial	$\mathbb{Z}_0^+$
$n!_m$	omnifactorial	$\mathbb{Z}$
$\lfloor n \rceil !_m$	Roman factorial of order m	$\mathbb{Z}$

Tbl. 1.3.1: Notation about the factorial variations

The factorial of order m is alternatively called **omnifactorial**, a new term presented in this paper. Technically, it refers to the most generalized version of the factorial examined in this study, but it is currently defined only for integers. Its domain will be expanded accordingly for each of the three papers that follow this one.

This system of notation is crucial as we will frequently encounter mixed expressions. For instance, the Roman factorial of order 2 includes the double factorial in its piece-wise definition, as detailed in Subsection 3.4. This notation might change slightly as the subsequent papers are written in the future.

In Part 1, there was no need to establish this notation, since we didn't examine factorials of higher orders. So, the factorial of order 1 as defined here coincides with the Roman factorial, and the latter symbolism is preserved for that paper only. References to definitions from Part 1 are written here using $n!_1$.

The next page is a summary of Part 1 and it contains everything that is relevant from the previous paper of this study.

1.4 Part 1 Summary

In summary, in Part 1:

- **Section 2**: We analyzed the traditional factorial and presented the Roman factorial.
- **Section 3**: We introduced a set of 5 *foundational functions* (*F.F.*).
- **Section 4**: We employed the *F.F.* shown in the previous section to rewrite the Roman factorial definition concisely, in its original formulation as well as its recursive form.
- **Section 5**: We introduced another set of 5 *F.F.*
- **Section 6**: We found non-recursive $\prod$-product definitions to describe the values outputted by the factorial of order 1. The expressions are either a rising or a falling product, split into two cases each (positive and negative integers). Additionally, we used all *F.F.* to unify the cases of these definitions, achieving universality.

In Sections 3 and 5 we defined the following *foundational functions*:

$$\delta(n) = \lfloor n \rfloor + 0.5 \qquad [-,+,+] \qquad \text{(Eq. 1.4.1)}$$

$$\theta(n) = \frac{\delta(n)}{|\delta(n)|} \qquad [-1,1,1] \qquad \text{(Eq. 1.4.2)}$$

$$\xi(n) = \frac{1+\theta(n)}{2} \qquad [0,1,1] \qquad \text{(Eq. 1.4.3)}$$

$$\xi'(n) = \frac{1-\theta(n)}{2} \qquad [1,0,0] \qquad \text{(Eq. 1.4.4)}$$

$$\eta(n) = \theta(n)^{-\lceil n \rceil - 1} \qquad [\pm 1,1,1] \qquad \text{(Eq. 1.4.5)}$$

$$\Theta(n) = \xi(n) \cdot \xi(-n) \qquad [0,1,0] \qquad \text{(Eq. 1.4.6)}$$

$$Q(n) = \theta(n) - \Theta(n) \qquad [-1,0,1] \qquad \text{(Eq. 1.4.7)}$$

$$Q'(n) = 1 - \Theta(n) \qquad [1,0,1] \qquad \text{(Eq. 1.4.8)}$$

$$\Psi(n) = n + \Theta(n) \qquad [n,1,n] \qquad \text{(Eq. 1.4.9)}$$

$$\Phi(n) = \Psi(n)^{\xi'(n)} \qquad [n,1,1] \qquad \text{(Eq. 1.4.10)}$$

1.5 Part 1 Results

In Section 4 we condensed the Roman factorial definition. It was originally defined as follows:

$$\lfloor n \rceil ! = \begin{cases} n! & , \ n \in \mathbb{Z}_0^+ \\ \dfrac{(-1)^{-n-1}}{(-n-1)!} & , \ n \in \mathbb{Z}^-, \end{cases} \qquad \text{(Eq. 1.5.1)}$$

where the factorial is defined recursively as

$$n! = n(n-1)! \,, \quad 0! = 1 \,, \quad n \in \mathbb{Z}^+ . \qquad \text{(Eq. 1.5.2)}$$

The generalized relationship is

$$\lfloor n \rceil ! = \eta(n) \cdot \big(|n| - \xi'(n)\big)!^{\,\theta(n)} \,, \quad n \in \mathbb{Z} \,, \qquad \text{(Eq. 1.5.3)}$$

where

$$n! = n(n-1)! \,, \quad 0! = 1 \,, \quad n \in \mathbb{Z}^+ . \qquad \text{(Eq. 1.5.2)}$$

Additionally, we rewrote the following doubly-recursive definition of the Roman factorial:

$$n!_1 = \begin{cases} n(n-1)!_1 & , \ n \in \mathbb{Z}^+ \\ \dfrac{(n+1)!_1}{n+1} & , \ n \in \mathbb{Z}^- \setminus \{-1\} \,, \end{cases} \qquad \text{(Eq. 1.5.4)}$$

where

$$0!_1 = (-1)!_1 = 1 \,. \qquad \text{(Eq. 1.5.5)}$$

The outcome of the generalization is as follows:

$$n!_1 = \big(n + \xi'(n)\big)^{\theta(n)} \big(n - \theta(n)\big)!_1 \,, \quad n \in \mathbb{Z} \setminus \{0, -1\} \,, \qquad \text{(Eq. 1.5.6)}$$

where

$$0!_1 = (-1)!_1 = 1 \,. \qquad \text{(Eq. 1.5.5)}$$

Lastly, in Section 6, we constructed these definitions:

$n!_1$	Rising product	Falling product
$n \in \mathbb{Z}_0^+$	$\prod_{k=1}^{n} k$	$\prod_{k=0}^{n-1} (n-k)$
$n \in \mathbb{Z}^-$	$n \cdot \prod_{k=1}^{-n} \frac{1}{-k}$	$n \cdot \prod_{k=0}^{-n-1} \frac{1}{n+k}$

Tbl. 1.5.1: Factorial of order 1 as a rising or falling product

The two definitions in the above table were consolidated into the formulations listed below:

$n!_1$	$n \in \mathbb{Z}$
Rising product	$\Phi(n) \cdot \prod_{k=1}^{\lvert n \rvert} \big(k\,\theta(n)\big)^{\theta(n)}$
Falling product	$\Phi(n) \cdot \prod_{k=0}^{\lvert n \rvert - 1} \big(n - k\,\theta(n)\big)^{\theta(n)}$

Tbl. 1.5.2: Factorial of order 1 as a rising or falling product (generalized)

In this part, we will examine the double factorial and follow a similar procedure as before. We will investigate an extension into negative integers, we will find recursive as well as non-recursive piece-wise definitions of the expanded double factorial and we will unify their cases. This process will be repeated for the triple factorial and in general for all factorials of higher orders, resulting in a universal expression that can be used to calculate any variation of the factorial presented so far.

2 Double factorial

2.1 Introduction

The double factorial [4], denoted as $n!!$, is a variation of the traditional factorial [3]. Specifically, $n!!$ represents the product of all the positive integers up to n that have the same parity (odd or even) as n, as shown below:

$$5!! = 5 \cdot 3 \cdot 1 = 15\,,$$
$$6!! = 6 \cdot 4 \cdot 2 = 48\,,$$
$$7!! = 7 \cdot 5 \cdot 3 \cdot 1 = 105\,.$$

Notice that depending on whether n is even or odd, the product terminates with either 2 or 1. Initially, this property complicates the construction of intuitive definitions for $n!!$, but it ultimately provides insights into how some extensions of the factorial behave.

Tbl. 2.1.1 presents the values of double factorials alongside traditional factorials for comparison:

n	0	1	2	3	4	5	6	7
$n!$	1	1	2	6	24	120	720	5040
$n!!$	1	1	2	3	8	15	48	105

Tbl. 2.1.1: Factorials and double factorials

This variation grows more slowly than the traditional factorial, as illustrated in the figure below[1]. This behavior is expected since the double factorial involves roughly half as many factors as the corresponding traditional factorial.

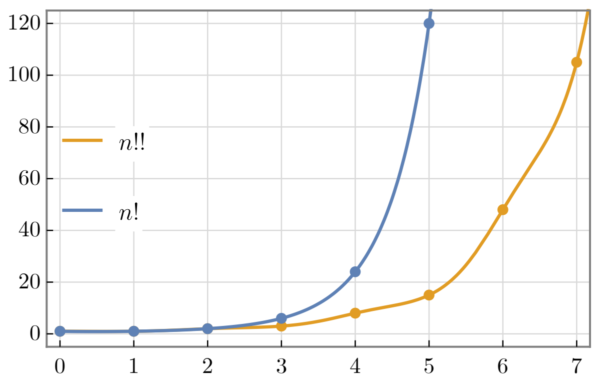

Fig. 2.1.1: Double Factorial

Although the double factorial has several properties that fall outside the scope of this paper, further exploration can be found in Addendum 10.2.

[1]The double factorial has a continuous extension based on the Gamma function, which was used to create Fig. 2.1.1. Further details can be found in Addendum 10.3.

To conclude this introduction, let's briefly explain why $0!! = 1!! = 1$ (as indicated in Tbl. 2.1.1) by recalling the approach we used to find $0!$ in Part 1:

$$3! = 3 \cdot 2 \cdot 1 = \frac{4 \cdot 3 \cdot 2 \cdot 1}{4} = \frac{4!}{4}\,,$$
$$2! = 2 \cdot 1 = \frac{3 \cdot 2 \cdot 1}{3} = \frac{3!}{3}\,,$$
$$1! = 1 = \frac{2 \cdot 1}{2} = \frac{2!}{2}\,,$$
$$0! = \frac{1}{1} = 1\,.$$

This reasoning can be applied to the double factorial, for both even and odd n. For odd n, we have:

$$7!! = 7 \cdot 5 \cdot 3 \cdot 1 = \frac{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}{9} = \frac{9!!}{9}\,,$$
$$5!! = 5 \cdot 3 \cdot 1 = \frac{7 \cdot 5 \cdot 3 \cdot 1}{7} = \frac{7!!}{7}\,,$$
$$3!! = 3 \cdot 1 = \frac{5 \cdot 3 \cdot 1}{5} = \frac{5!!}{5}\,,$$
$$1!! = 1 = \frac{3 \cdot 1}{3} = \frac{3!!}{3}\,.$$

Repeating this process once more gives $(-1)!! = 1$. Although negative double factorials are not the focus here, this is essentially how they will be defined later.

For even n we find $0!! = 1$:

$$6!! = 6 \cdot 4 \cdot 2 = \frac{8 \cdot 6 \cdot 4 \cdot 2}{8} = \frac{8!!}{8}\,,$$
$$4!! = 4 \cdot 2 = \frac{6 \cdot 4 \cdot 2}{6} = \frac{6!!}{6}\,,$$
$$2!! = 2 = \frac{4 \cdot 2}{4} = \frac{4!!}{4}\,,$$
$$\Rightarrow 0!! = \frac{2!!}{2} = 1\,.$$

This result is intuitive using this approach, but another reason $0!! = 1$ is that it represents the result of an empty product, which is inherently 1 (the multiplicative identity)[2].

Having demonstrated the behavior of the double factorial for $n \in \mathbb{Z}_0^+$, we can now proceed to define it formally[3]. The remainder of this section focuses on deriving three distinct definitions of the double factorial: we will achieve this by starting with the traditional factorial and applying modifications to its established definitions.

[2]This paper emphasizes intuitive explanations for results like $0!! = 1!! = 1$ rather than formal mathematical proofs in this context.

[3]The set denoted by $\mathbb{Z}_0^+$ is the set that includes all positive integers ($\mathbb{Z}^+$) and 0. In early versions of Part 1, the set $\mathbb{N}_0$ was used to denote the same set. However, we avoid this notation here for consistency. More about number sets can be found in Addendum 10.1.

2.2 Recursive definition

For the rest of this section, we will explore 3 definitions of $n!!$ for $n \in \mathbb{Z}_0^+$. We will begin by defining $n!!$ recursively, followed by establishing rising and falling product definitions. These last two definitions are analogous to those for the traditional factorial, as discussed in Part 1, and will be elaborated upon in the subsequent subsections[4].

The definition examined in this subsection is fundamentally simple, but we will focus on the foundational aspects. This approach will facilitate easier generalizations in later sections and provide valuable insights.

Let's begin by exploring the recursive nature of the double factorial through a few examples. We observe that all terms of the product, except the first, form another double factorial:

$$\begin{aligned} 5!! &= 5 \cdot 3 \cdot 1 = 5 \cdot (3 \cdot 1) = 5 \cdot 3!!\,, \\ 6!! &= 6 \cdot 4 \cdot 2 = 6 \cdot (4 \cdot 2) = 6 \cdot 4!!\,, \\ 7!! &= 7 \cdot 5 \cdot 3 \cdot 1 = 7 \cdot (5 \cdot 3 \cdot 1) = 7 \cdot 5!!\,, \\ 8!! &= 8 \cdot 6 \cdot 4 \cdot 2 = 8 \cdot (6 \cdot 4 \cdot 2) = 8 \cdot 6!!\,. \end{aligned}$$

This pattern resembles that of the traditional factorial, where $n!$ is related to $(n-1)!$ as follows:

$$n! = n(n-1)!\,, \quad n \in \mathbb{Z}^+. \qquad \text{(Eq. 2.2.1)}$$

From the examples above, as well as from Eq. 2.2.1 we can derive the recursive property of the double factorial. It is as follows:

$$n!! = n(n-2)!!\,, \quad n \in \mathbb{Z}^+ \setminus \{1\}\,, \qquad \text{(Eq. 2.2.2)}$$

but this equation does not completely describe a recursive definition, as we'll see shortly.

Notice that Eq. 2.2.2 is undefined for the smallest positive integer, $n = 1$. Evaluating this case yields:

$$1!! = 1(1-2)!! \Rightarrow 1 = (-1)!!\,,$$

which currently lies outside the domain of $n!!$.

To be well-defined, a recursive definition requires two components: a recursive relation and an initial condition. For example, consider the recursive definition of the traditional factorial:

$$n! = n(n-1)!\,, \quad 0! = 1\,, \quad n \in \mathbb{Z}^+. \qquad \text{(Eq. 1.5.2)}$$

This definition relates $n!$ with $(n-1)!$ and specifies $0! = 1$. In other words, any traditional factorial can be calculated using the preceding factorial, terminating at the base case $0! = 1$. Eq. 1.5.2 requires only one such base case, or "seed", as its domain consists of natural numbers and links consecutive integers.

[4]For more information on rising and falling products, including their interpretation for products involving reciprocals of integers, see Addendum 10.4.

For instance, calculating 4! using Eq. 1.5.2 proceeds as follows::

$$\begin{aligned} 4! &= 4 \cdot 3! \\ &= 4 \cdot (3 \cdot 2!) \\ &= 4 \cdot 3 \cdot (2 \cdot 1!) \\ &= 4 \cdot 3 \cdot 2 \cdot (1 \cdot 0!) \\ &= 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1 \\ &= 24\,. \end{aligned}$$

The following figure illustrates this calculation in a straightforward manner:

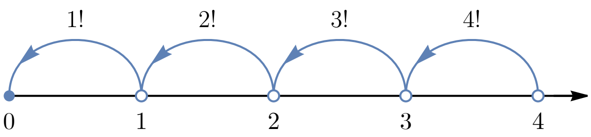

Fig. 2.2.1: Traditional factorial recursiveness

This figure shows the iterative process of calculating $4!$, starting with $4!$, then applying the recursive definition to find $3!$, followed by $2!$, $1!$, and finally $0!$. Once the base case $0!$ is reached, the process terminates.

Using Fig. 2.2.1 and the methodical calculation of traditional factorials described above, we see that only one base case is needed for their recursive definition.

Returning to the double factorial, let's perform a similar analysis. To calculate 5!! or 4!! using its recursive property, we proceed as follows:

$$\begin{aligned} 5!! &= 5 \cdot 3!! & \qquad 4!! &= 4 \cdot 2!! \\ &= 5 \cdot (3 \cdot 1!!) & &= 4 \cdot (2 \cdot 0!!) \\ &= 5 \cdot 3 \cdot 1 & &= 4 \cdot 2 \cdot 1 \\ &= 15\,, & &= 8\,. \end{aligned}$$

Note that $0!! = 1!! = 1$ is required for this calculation to be valid. Setting only one of these double factorials to 1 does not cover all cases, as illustrated in the following figure:

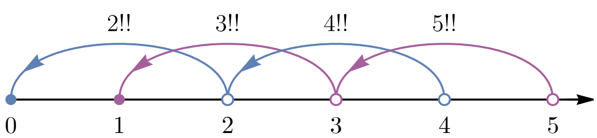

Fig. 2.2.2: Double factorial recursiveness

Thus, two seeds are needed for Eq. 2.2.2 to function correctly. We select the smallest non-negative integers, resulting in the following expression:

$$n!! = n(n-2)!!\,, \quad 0!! = 1!! = 1\,, \quad n \in \mathbb{Z}^+ \setminus \{1\}\,. \qquad \text{(Eq. 2.2.3)}$$

Eq. 2.2.3 is the recursive definition of the double factorial. In the next subsection, we will explore non-recursive definitions in the form of $\prod$-products.

2.3 Falling product definition

To derive the falling product definition of the double factorial, we seek a product expression with specific upper and lower limits and a formula for the multiplicand k.

We will begin by referencing the analogous definition of the traditional factorial to guide our process. Next, we will derive the expression for k and determine the number of factors in $n!!$.

Recall that a falling product has terms that progressively decrease, as detailed in Addendum 10.4. In Part 1, the falling product definition for the traditional factorial was established as follows:

$$n! = \prod_{k=0}^{n-1} (n-k)\,, \quad n \in \mathbb{Z}^+. \qquad \text{(Eq. 2.3.1)}$$

For example, the calculation of 5! is demonstrated below:

$$5! = \prod_{k=0}^{4} (5-k) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120\,.$$

In this case, the lower limit of the product is $k = 0$, ensuring that the factor n is included in $n!$. The product consists of n terms (starting from $k = 0$), which justifies the upper limit of $n - 1$.

Given that the double factorial inherently has half the number of factors as the regular factorial, our initial hypothesis is that $n!!$ contains $\frac{n}{2}$ terms. If we assume k starts from $k = 0$ again, the upper limit would be $\frac{n}{2} - 1$.

Additionally, note that the terms of $n!!$ in a falling product decrease by 2 at each step. This can be expressed using the product of terms $(n - 2k)$.

Based on these observations, we intuitively propose the following definition:

$$n!! = \prod_{k=0}^{\frac{n}{2}-1} (n-2k)\,, \quad n \in \mathbb{Z}^+. \qquad \text{(Eq. 2.3.2)}$$

However, this definition is flawed. Although it is valid for even n, the upper limit is not an integer for odd n, which makes the product undefined in such cases.

To correct this, we need to analyze the number of factors in odd double factorials. Let's first consider examples for even $n!!$:

$$\begin{aligned}
2!! &= 2 && \rightarrow 1 \text{ term} \\
4!! &= 4 \cdot 2 && \rightarrow 2 \text{ terms} \\
6!! &= 6 \cdot 4 \cdot 2 && \rightarrow 3 \text{ terms} \\
8!! &= 8 \cdot 6 \cdot 4 \cdot 2 && \rightarrow 4 \text{ terms} \\
&\vdots && \vdots \\
n!! &= n(n-2)(n-4)\cdots && \rightarrow \frac{n}{2} \text{ terms}\,.
\end{aligned}$$

Clearly, $n!!$ indeed has $\frac{n}{2}$ terms when n is even. In other words, the product of all even integers up to n consists of $\frac{n}{2}$ factors.

Now let's examine odd $n!!$:

$$\begin{aligned}
1!! &= 1 && \rightarrow 1 \text{ term} \\
3!! &= 3 \cdot 1 && \rightarrow 2 \text{ terms} \\
5!! &= 5 \cdot 3 \cdot 1 && \rightarrow 3 \text{ terms} \\
7!! &= 7 \cdot 5 \cdot 3 \cdot 1 && \rightarrow 4 \text{ terms} \\
&\vdots && \vdots \\
n!! &= n(n-2)(n-4)\cdots && \rightarrow \frac{n+1}{2} \text{ terms}\,.
\end{aligned}$$

For odd n, the product of all odd integers up to n contains $\frac{n+1}{2}$ terms. This fraction is always an integer because it describes odd n, so $n + 1$ is even and $\frac{n+1}{2}$ is an integer.

Let's gather what we have just observed in a table. So far, we have found the following conditions[5]:

$n!!$	# of factors
n is even	$\frac{n}{2}$
n is odd	$\frac{n+1}{2}$

Tbl. 2.3.1: Number of factors in $n!!$

We now seek a mathematical expression that covers both cases. There are more than one approaches to this problem, as will be briefly mentioned later in Subsection 3.4.

Here we will proceed with the ceiling function[6], or "rounding up". Notice that the quantity $\frac{n}{2}$ is a half-integer for odd n, and it is made an integer by adding $\frac{1}{2}$ to it. Effectively, this is rounding up to the next bigger integer.

When n is even, the ceiling function does not alter $\frac{n}{2}$ in any way since it is already an integer. Thus, we can express the number of factors in $n!!$ as:

$$\left\lceil \frac{n}{2} \right\rceil = \begin{cases} \dfrac{n}{2} & , \ n \text{ is even} \\[2ex] \dfrac{n+1}{2} & , \ n \text{ is odd}\,. \end{cases} \qquad \text{(Eq. 2.3.3)}$$

Hence, the double factorial can be expressed as a product with $\lceil \frac{n}{2} \rceil$ terms, leading to the following falling product definition:

$$n!! = \prod_{k=0}^{\lceil \frac{n}{2} \rceil - 1} (n-2k)\,, \quad n \in \mathbb{Z}^+. \qquad \text{(Eq. 2.3.4)}$$

We will next develop a similar definition for a rising product.

[5]The hashtag symbol "#" is an abbreviation and stands for the word "number".

[6]The ceiling function rounds up to the next integer, while the floor function rounds down. Additionally, the sawtooth function represents the fractional part of a number, denoted by $\{n\}$. For more details on the floor, ceiling, and sawtooth functions, refer to Addendum 10.5.

2.4 Rising product definition

In this subsection, we will analyze the rising product in a way similar to our earlier discussion on the falling product. First, let's recall the definition of the rising product for the traditional factorial:

$$n! = \prod_{k=1}^{n} k\,, \quad n \in \mathbb{Z}^{+}. \qquad \text{(Eq. 2.4.1)}$$

This is arguably the simplest $\prod$-product possible. Note that its limits differ from those of the falling product: the limits for $n!$ go from 1 to n, whereas the limits for the falling product range from 0 to $n-1$. This change only affects the counting of factors, not their total number.

For example, Eq. 2.4.1 is demonstrated below:

$$5! = \prod_{k=1}^{5} k = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120\,.$$

Now, let's adapt this definition to find a similar expression for the double factorial.

We start by adjusting the upper limit of the $\prod$-product to match the behavior of the double factorial. Therefore, the upper limit should be set to $\lceil \frac{n}{2} \rceil$, as determined earlier.

Additionally, we replace k with $2k$, because the factors of the double factorial increase by 2 with each step.

These changes give us the following definition:

$$n!! = \prod_{k=1}^{\lceil \frac{n}{2} \rceil} 2k\,, \quad n \in \mathbb{Z}^{+}. \qquad \text{(Eq. 2.4.2)}$$

This definition works for even n, but fails for odd integers. The following examples highlight this oversight:

$$6!! = \prod_{k=1}^{\lceil \frac{6}{2} \rceil} 2k = \prod_{k=1}^{3} 2k = 2 \cdot 4 \cdot 6 = 48\,,$$

$$5!! = \prod_{k=1}^{\lceil \frac{5}{2} \rceil} 2k = \prod_{k=1}^{3} 2k = 2 \cdot 4 \cdot 6 = 48\,.$$

It seems that for odd integers, the product should start with 1 instead of 2. To fix this, we need to subtract 1 from $2k$, resulting in:

$$n!! = \prod_{k=1}^{\lceil \frac{n}{2} \rceil} (2k-1)\,, \quad n \in \mathbb{Z}^{+}_{odd}\,. \qquad \text{(Eq. 2.4.3)}$$

So, the rising product definition of the double factorial must include a term to subtract from $2k$. For odd n this term is 1, and for even n it is 0, which matches the previous definition in Eq. 2.4.2.

We can use modular arithmetic to express this subtraction term[7]:

$$n \bmod 2 = \begin{cases} 0 & , \ n \text{ is even} \\ 1 & , \ n \text{ is odd}\,. \end{cases} \qquad \text{(Eq. 2.4.4)}$$

The expression $n \bmod 2$ gives the remainder of n when divided by 2. For instance, $3 \bmod 2 = 1$ and $4 \bmod 2 = 0$.

By subtracting 1 from $2k$, we can now evaluate $5!!$ correctly:

$$5!! = \prod_{k=1}^{\lceil \frac{5}{2} \rceil} (2k-1) = \prod_{k=1}^{3} (2k-1) = 1 \cdot 3 \cdot 5 = 15\,.$$

In conclusion, the rising product definition of the double factorial is:

$$n!! = \prod_{k=1}^{\lceil \frac{n}{2} \rceil} (2k - n \bmod 2)\,, \quad n \in \mathbb{Z}^{+}. \qquad \text{(Eq. 2.4.5)}$$

There are other ways to derive this definition without using modular arithmetic, but we chose this approach because it may help us generalize to higher orders of factorials in Section 5. Details about this choice will be discussed in the related subsection.

2.5 Summary

In Section 2, we explored the double factorial for positive integers and developed two $\prod$-product definitions based on the results from the traditional factorial in Part 1. These definitions are summarized in the table below:

$n!!$	$n \in \mathbb{Z}^{+}$
Falling product	$\prod_{k=0}^{\lceil \frac{n}{2} \rceil - 1} (n-2k)$
Rising product	$\prod_{k=1}^{\lceil \frac{n}{2} \rceil} (2k - n \bmod 2)$

Tbl. 2.5.1: Double factorial expressed as a rising or falling product for positive integers

We also established the following recursive definition for double factorials:

$$n!! = n(n-2)!!\,, \quad 0!! = 1!! = 1\,, \quad n \in \mathbb{Z}^{+} \setminus \{1\}\,. \qquad \text{(Eq. 2.2.3)}$$

In the next section, we will look for ways to extend the double factorial to negative integers and suggest definitions similar to those discussed here. The concepts from Section 2 will be crucial for these generalizations, but any aspects that are not fully understood will be revisited as needed.

[7]More about modular arithmetic in Addendum 10.6.

3 Negative double factorials

3.1 Introduction

Negative double factorials are not commonly written about, leaving a gap in our understanding. In this section, we will intuitively define it by extrapolating from negative odd double factorials and comparing them to the Roman factorial.

We will start by determining values for negative odd $n!_2$ and formulating a corresponding definition. Next, we will address negative even double factorials and derive a similar formula. After establishing values and definitions for all negative $n!_2$, we will summarize our findings with four distinct definitions.

The phrase "negative double factorials" does not align well with the notation established earlier in this paper, but in this case it is preferred compared to "negative factorials of order 2" because it is more concise. It is a term used only in this section.

3.2 Negative odd double factorials

Let's start with the recursive relationship for the double factorial, which connects $n!!$ with $(n-2)!!$ as shown here:

$$n!! = n(n-2)!!\,, \quad 0!! = 1!! = 1\,, \quad n \in \mathbb{Z}^+ \setminus \{1\}\,. \qquad \text{(Eq. 2.2.3)}$$

This formula helps find the values of larger $n!!$ when smaller values are known. However, for negative double factorials, we need a formula that works in the opposite direction.

By replacing n with $n+2$ in Eq. 2.2.3 and solving for n, we obtain a similar expression:

$$(n+2)!! = (n+2)n!! \Rightarrow n!! = \frac{(n+2)!!}{n+2}\,, \quad n \in \mathbb{Z}_0^+. \qquad \text{(Eq. 3.2.1)}$$

This formula relates $n!!$ to $(n+2)!!$, rather than $(n-2)!!$. It would be useful if we ever wanted to find a double factorial when only knowing a bigger one. For example:

$$6!! = \frac{(6+2)!!}{6+2} = \frac{8!!}{8} = \frac{8\cdot 6\cdot 4\cdot 2}{8} = 6\cdot 4\cdot 2 = 48\,.$$

From here on forward, we will use the notation $n!_2$ to refer to double factorials in general, and $n!_{(2)}$ to replace $n!!$, as it's only defined for positive integers.

We know that Eq. 3.2.1 works for non-negative integers, and has initial values (or "seeds"): $0!_{(2)} = 1!_{(2)} = 1$. Let's try to see what would happen if we tried to find $(-1)!_2$ using it, as a test[8]:

$$(-1)!_2 = \frac{(-1+2)!_2}{-1+2} = \frac{1!_2}{1} = 1\,.$$

[8]This direct manner of attempting to find negative double factorials is not mathematically proper, because its recursive definition does not cover this domain. Rather, it answers the question: "What value should $(-1)!_2$ have in order for $1!_2$ to be found recursively?".

This method avoids the division by 0 problem we found with the Roman factorial, where we also tried to explore negative factorials by expanding their recursive relationship. The value of $(-1)!_1$ was undefined due to division by 0, so all other negative integer factorials were impossible to find since they were directly related to $(-1)!_1$.

However, in the case of the double factorial, $(-1)!_2$ can be determined without any division by 0. This provides insight into what negative odd $n!_2$ values can be, something not naturally derivable in the traditional factorial.

Having found a value for $(-1)!_2$, we can find $(-3)!_2$ and other negative odd double factorials using the same method. Here's the recursive process for more negative values:

$$(-3)!_2 = \frac{(-3+2)!_2}{-3+2} = \frac{(-1)!_2}{-1} = -1\,,$$

$$(-5)!_2 = \frac{(-5+2)!_2}{-5+2} = \frac{(-3)!_2}{-3} = \frac{1}{3}\,,$$

$$\vdots$$

We can now define a piece-wise expression for odd double factorials by combining the recursive formulas for positive and negative n. This definition starts with $1!_2 = 1$ and all other values are derived from essentially the same relationship.

The piece-wise definition is:

$$n!_2 = \begin{cases} n(n-2)!_2 & ,\ n \in \mathbb{Z}^+_{odd} \setminus \{1\} \\ 1 & ,\ n = 1 \\ \dfrac{(n+2)!_2}{n+2} & ,\ n \in \mathbb{Z}^-_{odd}\,. \end{cases} \qquad \text{(Eq. 3.2.2)}$$

Notice how close this definition is in comparison to the corresponding one about the Roman factorial. Its recursive definition, as found in Part 1, is:

$$n!_1 = \begin{cases} n(n-1)!_1 & ,\ n \in \mathbb{Z}^+ \\ 1 & ,\ n = \{0, -1\} \\ \dfrac{(n+1)!_1}{n+1} & ,\ n \in \mathbb{Z}^- \setminus \{-1\}\,. \end{cases} \qquad \text{(Eq. 1.5.4)}$$

The Roman factorial is defined for all integers, while Eq. 3.2.2 only applies to odd numbers. We will set aside the similarities to the Roman factorial however, since this approach helps us build definitions from scratch and avoid potential oversights.

Now, let's compile a table showing odd double factorials, including some positive values:

n	-9	-7	-5	-3	-1	1	3	5
$n!_2$	$1/105$	$-1/15$	$1/3$	-1	1	1	3	15

Tbl. 3.2.1: Odd double factorials

Two patterns emerge from Tbl. 3.2.1. First, negative odd double factorials have alternating signs. Second, they are the reciprocals of positive double factorials, with an offset. Let's update the table to highlight these features:

n	-9	-7	-5	-3
$n!_2$	$(+1)/7!_{(2)}$	$(-1)/5!_{(2)}$	$(+1)/3!_{(2)}$	$(-1)/1!_{(2)}$

Tbl. 3.2.2: Negative odd double factorials

To simplify, we excluded $(-1)!_2$ from this table because it does not fit the highlighted pattern.

Since negative odd $n!_2$ are fractions, the simplest definition we can form is:

$$n!_2 = \frac{f(n)}{g(n)}\,, \quad n \in \mathbb{Z}^-_{odd} \setminus \{-1\}\,, \qquad \text{(Eq. 3.2.3)}$$

where $f(n)$, $g(n)$ are arbitrary functions of n which will be found shortly.

To clarify these functions, let's update Tbl. 3.2.2 so that their outputs are clearer:

n	-9	-7	-5	-3
$n!_2$	$(+1)/7!_{(2)}$	$(-1)/5!_{(2)}$	$(+1)/3!_{(2)}$	$(-1)/1!_{(2)}$
$f(n)$	$+1$	-1	$+1$	-1
$g(n)$	$7!_2$	$5!_2$	$3!_2$	$1!_2$

Tbl. 3.2.3: Negative odd double factorials in terms of $f(n)$ and $g(n)$

The function $g(n)$ is straightforward. It can be expressed as:

$$g(n) = (-n-2)!_{(2)}\,, \quad n \in \mathbb{Z}^-_{odd} \setminus \{-1\}\,. \quad \text{(Eq. 3.2.4)}$$

Therefore, negative odd double factorials are defined as:

$$n!_2 = \frac{f(n)}{(-n-2)!_{(2)}}\,, \quad n \in \mathbb{Z}^-_{odd} \setminus \{-1\}\,, \quad \text{(Eq. 3.2.5)}$$

where

$$f(n) = \begin{cases} +1\,, & n \in \{-5, -9, -13 \ldots\} \\ -1\,, & n \in \{-3, -7, -11 \ldots\}\,. \end{cases} \qquad \text{(Eq. 3.2.6)}$$

Next, we need to find the analytical expression for $f(n)$. Since n is odd, it can be written as $n = 2k+1$, where $k \in \mathbb{Z}$. We will first find $f(k)$ and then return to $f(n)$. This approach will simplify relationships moving forward, and it will be applied again for factorials of higher orders in Section 6.

Let's create another table showing n, k, and $f(k)$ for some negative odd integers:

n	-9	-7	-5	-3
k	-5	-4	-3	-2
$f(k)$	$+1$	-1	$+1$	-1

Tbl. 3.2.4: The function $f(k)$ in negative odd $n!_2$

It seems that we need a formula for k that outputs these alternating signs.

An intuitive first choice is $(-1)^k$, the simplest sign-changing function. Let's test it into the table:

k	-5	-4	-3	-2
$f(k)$	$+1$	-1	$+1$	-1
$(-1)^k$	-1	$+1$	-1	$+1$

Tbl. 3.2.5: The function $(-1)^k$ in negative odd $n!_2$

It appears that $(-1)^k$ produces the opposite signs of what we need, so we will adjust by multiplying it by (-1), resulting in $(-1)(-1)^k = (-1)^{k+1}$:

k	-5	-4	-3	-2
$f(k)$	$+1$	-1	$+1$	-1
$(-1)^{k+1}$	$+1$	-1	$+1$	-1

Tbl. 3.2.6: The function $(-1)^{k+1}$ in negative odd $n!_2$

Indeed, this last expression for $f(k)$ outputs the desired signs correctly. Therefore, we have:

$$f(k) = (-1)^{k+1}\,, \quad k \in \mathbb{Z}^- \setminus \{-1\}\,. \quad \text{(Eq. 3.2.7)}$$

We have found the expression for $f(k)$ we were looking for, but initially we set it in terms of n. Reverting the exponent $k+1$ back to the variable n, we have:

$$k + 1 = \frac{n-1}{2} + 1 = \frac{n+1}{2}\,.$$

Thus, $f(n)$ is:

$$f(n) = (-1)^{\frac{n+1}{2}}\,, \quad k \in \mathbb{Z}^-_{odd} \setminus \{-1\}\,. \quad \text{(Eq. 3.2.8)}$$

Conclusively, we have built the formula that describes negative odd double factorials. It has the form:

$$n!_2 = \frac{(-1)^{\frac{n+1}{2}}}{(-n-2)!_{(2)}}\,, \quad n \in \mathbb{Z}^-_{odd} \setminus \{-1\}\,, \quad \text{(Eq. 3.2.9)}$$

where

$$n!_{(2)} = n(n-2)!_{(2)}\,, \quad 0!_{(2)} = 1!_{(2)} = 1\,, \quad n \in \mathbb{Z}^+ \setminus \{1\}\,. \qquad \text{(Eq. 2.2.3)}$$

In the next subsection, we will determine values for negative even double factorials and create a similar definition that includes all integers.

3.3 Negative even double factorials

The expression for negative odd double factorials in Eq. 3.2.9 bears a close resemblance to the Roman factorial:

$$\lfloor n \rceil ! = \frac{(-1)^{-n-1}}{(-n-1)!}\,, \quad n \in \mathbb{Z}^{-}, \qquad \text{(Eq. 1.5.1)}$$

where

$$n! = n(n-1)!\,, \quad 0! = 1\,, \quad n \in \mathbb{Z}^{+}. \qquad \text{(Eq. 1.5.2)}$$

Since negative odd double factorials resemble Roman factorials, we can use this similarity to define negative even $n!_2$. After achieving that, we will unite the definitions for negative odd and even $n!_2$ to a single expression, that encompasses all negative integers.

Let's now consider negative even double factorials. Unlike odd $n!_2$, we cannot derive values for these factorials through a recursive approach. Attempting to use the positive recursion formula Eq. 3.2.1 to find $(-2)!_2$, results in division by 0:

$$(-2)!_2 = \frac{(-2+2)!_2}{-2+2} = \frac{0!_2}{0} = \frac{1}{0}\,.$$

This problem is analogous to the undefined value of $\lfloor -1 \rceil !$. The Roman factorial resolved this by setting $\lfloor -1 \rceil ! = 1$ and recursively calculating other negative factorials.

We could follow a similar approach here. In fact, we will eventually define $(-2)!_2 = 1$ but doing this as a first step seems arbitrary and counter-intuitive.

Instead, consider the following: the Roman factorial for $n \in \mathbb{Z}^{-}$ is a product of reciprocals of negative integers. As a reminder, it is shown below[9]:

$$(-5)!_1 = (-5) \cdot \frac{1}{-5} \cdot \frac{1}{-4} \cdot \frac{1}{-3} \cdot \frac{1}{-2} \cdot \frac{1}{-1} = \frac{1}{24}\,.$$

Let's make a comparison with negative odd double factorials. As we will see in a later subsection, they can be thought of as a similar product:

$$(-9)!_2 = (-9) \cdot \frac{1}{-9} \cdot \frac{1}{-7} \cdot \frac{1}{-5} \cdot \frac{1}{-3} \cdot \frac{1}{-1} = \frac{1}{105}\,.$$

Given the overlap between these two concepts, we will aim to define negative even $n!_2$ in such a way that it follows the same logic.

For instance, we would want $(-10)!_2$ to be found using this exact product:

$$(-10)!_2 = (-10) \cdot \frac{1}{-10} \cdot \frac{1}{-8} \cdot \frac{1}{-6} \cdot \frac{1}{-4} \cdot \frac{1}{-2} = \frac{1}{384}\,.$$

Notice that the product avoids to end with $1/0$, because that is undefined. The logic behind it is similar to why $(-5)!_1$ in the example above does not terminate with the same undefined fraction.

[9] The left-most term cancels out with $-1/5$, but it is retained because this is how $n!_1$ was defined in Eq. 3.6.1 as a falling product. The same logic is applied to the double factorial.

By making the assumption that negative even $n!_2$ should follow this pattern, we can extrapolate this logic to other values as well:

$$(-8)!_2 = (-8) \cdot \frac{1}{-8} \cdot \frac{1}{-6} \cdot \frac{1}{-4} \cdot \frac{1}{-2} = -\frac{1}{48}\,,$$

$$(-6)!_2 = (-6) \cdot \frac{1}{-6} \cdot \frac{1}{-4} \cdot \frac{1}{-2} = \frac{1}{8}\,,$$

$$(-4)!_2 = (-4) \cdot \frac{1}{-4} \cdot \frac{1}{-2} = -\frac{1}{2}\,,$$

$$(-2)!_2 = (-2) \cdot \frac{1}{-2} = 1\,.$$

Thus, by following this approach of generalizing from negative odd $n!_2$, we end up with the following values for negative even double factorials:

n	-12	-10	-8	-6	-4	-2
$n!_2$	$-1/3840$	$1/384$	$-1/48$	$1/8$	$-1/2$	1

Tbl. 3.3.1: Negative even double factorials

Before we proceed, we have to remark on two key points. First, $(-2)!_2 = 1$ naturally follows from the assumption that negative even double factorials should behave similarly to their odd counterparts, given that this behavior is as shown in the examples of $(-9)!_2$ and $(-10)!_2$.

Secondly, the recurrence relation from Eq. 3.2.1 applies to the values in Tbl. 3.3.1. For example:

$$(-4)!_2 = \frac{(-2+2)!_2}{-4+2} = \frac{(-2)!_2}{-2} = -\frac{1}{2}\,.$$

Now, it can be expressed as a recursive definition:

$$n!_2 = \frac{(n+2)!_2}{n+2}\,, \quad n \in \mathbb{Z}^{-} \setminus \{-1, -2\}\,, \qquad \text{(Eq. 3.3.1)}$$

where

$$(-1)!_2 = (-2)!_2 = 1\,. \qquad \text{(Eq. 3.3.2)}$$

This behavior is illustrated in the next figure:

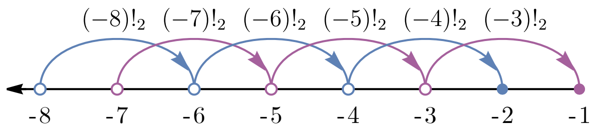

Fig. 3.3.1: Negative $n!_2$ recursiveness

Let's now find a definition that describes negative even $n!_2$. To start with, we notice that the definition should look similar to that of Eq. 3.2.9, since these factorials are also fractions with alternating signs and double factorials in the denominator.

It will be helpful to continue as before, by considering negative even double factorials as a ratio of two functions:

$$n!_2 = \frac{f(n)}{g(n)}\ , \quad n \in \mathbb{Z}^-_{even}\ , \qquad \text{(Eq. 3.3.3)}$$

where $f(n),\ g(n)$ are not necessarily the same functions as before.

The function $g(n)$ coincidentally, is the same as previously. It is shown in the next table:

n	-10	-8	-6	-4
$n!_2$	$^{(+1)}/_{8!_{(2)}}$	$^{(-1)}/_{6!_{(2)}}$	$^{(+1)}/_{4!_{(2)}}$	$^{(-1)}/_{2!_{(2)}}$
$f(n)$	$+1$	-1	$+1$	-1
$g(n)$	$8!_2$	$6!_2$	$4!_2$	$2!_2$

Tbl. 3.3.2: Negative even double factorials in terms of $f(n)$ and $g(n)$

We derive again that

$$g(n) = (-n-2)!_2\ , \quad n \in \mathbb{Z}^-_{even}\ . \qquad \text{(Eq. 3.3.4)}$$

Now, we will figure out the form of $f(n)$, or more precisely, $f(k)$. Let's create another table that that shows the values for k where $n = 2k$ along with a row for $f(k)$, which represents the alternating signs:

n	-10	-8	-6	-4
k	-5	-4	-3	-2
$f(k)$	$+1$	-1	$+1$	-1

Tbl. 3.3.3: The function $f(k)$ in negative even $n!_2$

Just like before, we can try using $(-1)^k$ as the expression for the alternating signs, but we quickly see that it outputs the opposite signs of what we want. To fix this, we can use $(-1)^{k+1}$ instead:

k	-5	-4	-3	-2
$f(k)$	$+1$	-1	$+1$	-1
$(-1)^{k+1}$	$+1$	-1	$+1$	-1

Tbl. 3.3.4: The function $(-1)^{k+1}$ in negative even double factorials

It looks like we've found the same expression for $f(k)$ as we did for the odd $n!_2$. However, in the previous case, k represented odd numbers, while now it stands for even numbers. If we switch back to using n instead of k, we get:

$$n = 2k \Rightarrow k+1 = \frac{n}{2} + 1\ .$$

So, the formula for negative even $n!_2$ is:

$$n!_2 = \frac{(-1)^{\frac{n}{2}+1}}{(-n-2)!_{(2)}}\ , \quad n \in \mathbb{Z}^-_{even}\ , \qquad \text{(Eq. 3.3.5)}$$

where

$$n!_{(2)} = n(n-2)!_{(2)}\ ,\ \ 0!_{(2)} = 1!_{(2)} = 1\ ,\ \ n \in \mathbb{Z}^+\backslash\{1\}\ . \qquad \text{(Eq. 2.2.3)}$$

Our next task is to combine the formulas for negative even and odd double factorials, which we will do right now.

3.4 Roman-like definition

In this subsection, we will combine the two definitions we have just derived into a single expression:

$$\lfloor n \rceil!_2 = \begin{cases} \dfrac{(-1)^{\frac{n+1}{2}}}{(-n-2)!_{(2)}} & ,\ n \in \mathbb{Z}^-_{odd} \setminus \{-1\} \\[2ex] \dfrac{(-1)^{\frac{n}{2}+1}}{(-n-2)!_{(2)}} & ,\ n \in \mathbb{Z}^-_{even}\ . \end{cases} \qquad \text{(Eq. 3.4.1)}$$

where

$$n!_{(2)} = n(n-2)!_{(2)}\ ,\ \ 0!_{(2)} = 1!_{(2)} = 1\ ,\ \ n \in \mathbb{Z}^+\backslash\{1\}\ . \qquad \text{(Eq. 2.2.3)}$$

But first, let's simplify the domains. As discussed later in Subsection 3.5, the recursive definition for positive double factorials can include the case when $n = -1$. With this, we can rewrite the equation as:

$$\lfloor n \rceil!_2 = \begin{cases} \dfrac{(-1)^{\frac{n+1}{2}}}{(-n-2)!_{(2)}} & ,\ n \in \mathbb{Z}^-_{odd} \\[2ex] \dfrac{(-1)^{\frac{n}{2}+1}}{(-n-2)!_{(2)}} & ,\ n \in \mathbb{Z}^-_{even}\ . \end{cases} \qquad \text{(Eq. 3.4.2)}$$

where

$$n!_{(2)} = n(n-2)!_{(2)}\ ,\ \ 0!_{(2)} = (-1)!_{(2)} = 1\ ,\ \ n \in \mathbb{Z}^+. \qquad \text{(Eq. 3.4.3)}$$

Since these resemble the Roman factorial definition, we'll call the unified expression a "Roman-like" definition of negative double factorials. We'll use the symbol $\lfloor n \rceil!_2$ to indicate this specific form, even though it describes the same values as other definitions we'll encounter later, which use $n!_2$.

Now, let's examine the cases in the above definition. They only differ in the exponent of the numerator, so we'll express it as a function of n and find its form:

$$f(n) = \begin{cases} \dfrac{n+1}{2} & ,\ n \in \mathbb{Z}^-_{odd} \\[2ex] \dfrac{n}{2} + 1 & ,\ n \in \mathbb{Z}^-_{even}\ . \end{cases} \qquad \text{(Eq. 3.4.4)}$$

There are many ways to unite these terms. For example, one method is to define a $F.F.$ that outputs

0 for odd n and $\frac{1}{2}$ for even n. This approach would work because Eq. 3.4.2 can be written as:

$$f(n) = \begin{cases} \dfrac{n+1}{2} + 0 & , \; n \in \mathbb{Z}^-_{odd} \\ \dfrac{n+1}{2} + \dfrac{1}{2} & , \; n \in \mathbb{Z}^-_{even} \, . \end{cases} \qquad \text{(Eq. 3.4.5)}$$

You can verify for yourself that the expressions $\frac{1}{2} - \{\frac{n}{2}\}$ and $\frac{1}{2} \cdot \Theta(\{\frac{n}{2}\})$ work as intended, however this method is too specific and does not lead to a simple expression going forward.

Instead, we can use the ceiling function. Notice that $\frac{n+1}{2}$ is an integer for odd n and half-integer for even n. Adding $\frac{1}{2}$ to it, makes it an integer for even n:

$$\left(\frac{n+1}{2}\right) \in \mathbb{Z} \quad \text{if } n \text{ is odd}\,,$$

$$\left(\frac{n+1}{2}\right) \notin \mathbb{Z} \quad \text{if } n \text{ is even}\,,$$

$$\left(\frac{n+1}{2} + \frac{1}{2}\right) = \left(\frac{n}{2} + 1\right) \in \mathbb{Z} \quad \text{if } n \text{ is even}\,.$$

The behavior can be replicated by rounding up $\frac{n+1}{2}$ to the next larger integer. This results in $\lceil \frac{n+1}{2} \rceil$, which is identical to $\frac{n+1}{2}$ for odd n (since they are both integers):

$$\left\lceil \frac{n+1}{2} \right\rceil = \begin{cases} \dfrac{n+1}{2} & , \; n \in \mathbb{Z}^-_{odd} \\ \dfrac{n}{2} + 1 & , \; n \in \mathbb{Z}^-_{even} \, . \end{cases} \qquad \text{(Eq. 3.4.6)}$$

Thus, the function we are looking for is $f(n) = \lceil \frac{n+1}{2} \rceil$, and substituting that in the piece-wise definition results in:

$$\lfloor n \rceil !_2 = \frac{(-1)^{\lceil \frac{n+1}{2} \rceil}}{(-n-2)!_{(2)}} \,, \quad n \in \mathbb{Z}^-, \qquad \text{(Eq. 3.4.7)}$$

where

$$n!_{(2)} = n(n-2)!_{(2)} \,, \quad 0!_{(2)} = (-1)!_{(2)} = 1 \,, \quad n \in \mathbb{Z}^+ . \qquad \text{(Eq. 3.4.3)}$$

Finally, combining Eq. 3.4.7 with a case for double factorials, akin to the Roman factorial formulation, gives us the Roman-like definition for the double factorial:

$$\lfloor n \rceil !_2 = \begin{cases} n!_{(2)} & , \; n \in \mathbb{Z}^+_0 \\ \dfrac{(-1)^{\lceil \frac{n+1}{2} \rceil}}{(-n-2)!_{(2)}} & , \; n \in \mathbb{Z}^-, \end{cases} \qquad \text{(Eq. 3.4.8)}$$

where

$$n!_{(2)} = n(n-2)!_{(2)} \,, \quad 0!_{(2)} = (-1)!_{(2)} = 1 \,, \quad n \in \mathbb{Z}^+ . \qquad \text{(Eq. 3.4.3)}$$

In Section 4 we will unify its two parts, using the *generalization process*.

3.5 Recursive definition

Here, we aim to define double factorials recursively for all $n \in \mathbb{Z}$. Previously, we've established recursive definitions for $n \in \mathbb{Z}^+ \setminus \{1\}$ (Eq. 2.2.3) and for $n \in \mathbb{Z}^- \setminus \{-1, -2\}$ (Eq. 3.3.1). Our goal here is to unify these into a single, intuitive recursive definition, similar to the recursive definition of the Roman factorial explored in Part 1.

To provide a clear objective, let's first recall the doubly-recursive definition of the Roman factorial:

$$n!_1 = \begin{cases} n(n-1)!_1 & , \; n \in \mathbb{Z}^+ \\ \dfrac{(n+1)!_1}{n+1} & , \; n \in \mathbb{Z}^- \setminus \{-1\} \,, \end{cases} \qquad \text{(Eq. 1.5.4)}$$

with the initial conditions

$$0!_1 = (-1)!_1 = 1 \,. \qquad \text{(Eq. 1.5.5)}$$

This is termed a "doubly-recursive" definition because it uses two different recursive relations, each working in opposite directions for positive and negative integers.

To better grasp this concept, consider the illustration below, which visualizes the recursiveness of the Roman factorial:

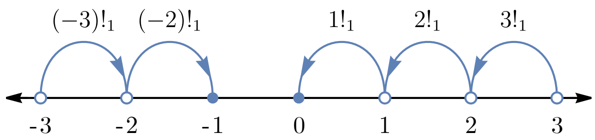

Fig. 3.5.1: Roman factorial recursiveness

The figure's right half illustrates the recursive behavior for positive integers, while the left half depicts the recursive behavior for negative integers. Positive integer factorials are computed starting from the seed $n = 0$, while negative integer factorials start from $n = -1$. Notably, there is no connection between these two seeds, since the recursive definitions do not evaluate directly at these points without encountering a division by 0.

Having revisited the recursive nature of the Roman factorial, we now turn back to the double factorial. To construct a similar recursive definition, let's recall the established definition for positive integers:

$$n!_{(2)} = n(n-2)!_{(2)} \,, \quad 0!_{(2)} = 1!_{(2)} = 1 \,, \quad n \in \mathbb{Z}^+ \setminus \{1\} . \qquad \text{(Eq. 2.2.3)}$$

And also the expression for negative integers:

$$n!_2 = \frac{(n+2)!_2}{n+2} \,, \quad n \in \mathbb{Z}^- \setminus \{-1, -2\} \,, \qquad \text{(Eq. 3.3.1)}$$

where

$$(-1)!_2 = (-2)!_2 = 1 \,. \qquad \text{(Eq. 3.3.2)}$$

We can combine these definitions right now into a unified expression, similar to Eq. 1.5.4. Let's construct it now to see where this approach leads us:

$$n!_2 = \begin{cases} n(n-2)!_2 & , \ n \in \mathbb{Z}^+ \setminus \{1\} \\ \dfrac{(n+2)!_2}{n+2} & , \ n \in \mathbb{Z}^- \setminus \{-1, -2\}, \end{cases} \qquad \text{(Eq. 3.5.1)}$$

where

$$1!_2 = 0!_2 = (-1)!_2 = (-2)!_2 = 1\,. \qquad \text{(Eq. 3.5.2)}$$

However, while the equations in this unified form are correct, the domain and initial conditions need refinement. There are four seeds, but one is redundant and can be eliminated.

In Subsections 2.1 and 2.2, we observed that the recursive definition for the double factorial is valid for -1. Let's revisit this point:

$$1!_2 = 1(1-2)!_2 \Rightarrow 1 = (-1)!_2\,.$$

Given this, we can rewrite Eq. 2.2.3 in this way:

$$n!_{(2)} = n(n-2)!_{(2)}\,, \ \ 0!_{(2)} = (-1)!_{(2)} = 1\,, \ \ n \in \mathbb{Z}^+. \qquad \text{(Eq. 3.5.3)}$$

This adjustment allows the positive and negative recursive definitions to share the seed -1 in common, thereby reducing the number of necessary initial conditions for the recursive definition of the double factorial.

This concept can be visualized in the following illustration:

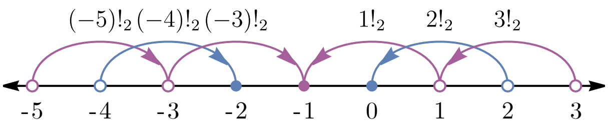

Fig. 3.5.2: Double factorial recursiveness with positive and negative integers

The figure shows how the recursion operates across all integers. Arrows indicate the direction in which the values of $n!_2$ are computed, either increasing or decreasing by 2, based on the initial conditions.

In summary, we arrive at the following recursive definition for the double factorial:

$$n!_2 = \begin{cases} n(n-2)!_2 & , \ n \in \mathbb{Z}^+ \\ \dfrac{(n+2)!_2}{n+2} & , \ n \in \mathbb{Z}^- \setminus \{-1, -2\}, \end{cases} \qquad \text{(Eq. 3.5.4)}$$

where

$$0!_2 = (-1)!_2 = (-2)!_2 = 1\,. \qquad \text{(Eq. 3.5.5)}$$

This represents the second piece-wise expression of the double factorial for all integers. In the rest of Section 3, we will explore two additional definitions which are not recursive.

3.6 Falling product definition

This subsection, along with the next, focuses on modifying the definitions of the Roman factorial to derive similar ones that describe the double factorial.

In Part 1, we defined the falling product definition of the Roman factorial as

$$n!_1 = n \cdot \prod_{k=0}^{-n-1} \frac{1}{n+k}\,, \quad n \in \mathbb{Z}^-. \qquad \text{(Eq. 3.6.1)}$$

Here's an example showing that the terms in the $\prod$-product decrease progressively[10]:

$$(-5)!_1 = (-5) \cdot \frac{1}{-5} \cdot \frac{1}{-4} \cdot \frac{1}{-3} \cdot \frac{1}{-2} \cdot \frac{1}{-1} = \frac{1}{24}\,.$$

We now need to adapt this formula for the double factorial.

To do this, we must determine two things: the limits of the $\prod$-product and the expression for each term inside it. This is straightforward since similar steps have been used before.

First, let's find examples of negative double factorials as falling products. These examples will help us understand the definition we need.

Let's start by expanding $(-9)!_2$ and $(-10)!_2$ in the following way, so that their factors decrease in order:

$$(-9)!_2 = (-9) \cdot \frac{1}{-9} \cdot \frac{1}{-7} \cdot \frac{1}{-5} \cdot \frac{1}{-3} \cdot \frac{1}{-1} = \frac{1}{105}\,,$$

$$(-10)!_2 = (-10) \cdot \frac{1}{-10} \cdot \frac{1}{-8} \cdot \frac{1}{-6} \cdot \frac{1}{-4} \cdot \frac{1}{-2} = \frac{1}{384}\,.$$

These results suggest that the terms of $n!_2$ decrease by 2, since $n!_2$ contains every other integer in its product expansion. This leads us to:

$$\frac{1}{n+k} \to \frac{1}{n+2k}\,.$$

Indeed, this expression describes the fractions in the example above.

Next, we need to set the limits for the product. Since the double factorial has half as many factors as the traditional factorial (rounding up for odd n), we adjust the upper limit of the product from $-n-1$ to $\lceil \frac{-n}{2} \rceil - 1$ so that it matches the expression in Subsection 2.3. Thus, we get:

$$n!_2 = n \cdot \prod_{k=0}^{\lceil \frac{-n}{2} \rceil - 1} \frac{1}{n+2k}\,, \quad n \in \mathbb{Z}^-. \qquad \text{(Eq. 3.6.2)}$$

Let's test the validity of this definition for various numbers. For even n, we have:

$$(-8)!_2 = (-8) \cdot \prod_{k=0}^{\lceil \frac{8}{2} \rceil - 1} \frac{1}{-8+2k} = (-8) \cdot \prod_{k=0}^{3} \frac{1}{-8+2k}$$
$$= (-8) \cdot \frac{1}{-8} \cdot \frac{1}{-6} \cdot \frac{1}{-4} \cdot \frac{1}{-2} = -\frac{1}{48}\,.$$

[10]Specifically, the absolute value of the denominators decrease. More about this concept in Addendum 10.4.

And for odd n:

$$(-9)!_2 = (-9) \cdot \prod_{k=0}^{\lceil \frac{9}{2} \rceil - 1} \frac{1}{-9+2k} = (-9) \cdot \prod_{k=0}^{4} \frac{1}{-9+2k}$$
$$= (-9) \cdot \frac{1}{-9} \cdot \frac{1}{-7} \cdot \frac{1}{-5} \cdot \frac{1}{-3} \cdot \frac{1}{-1} = \frac{1}{105}\,.$$

These results confirm that Eq. 3.6.2 accurately describes negative double factorials as a falling product. Rather than discovering a new expression from scratch, this time we modified a previous definition to our needs.

3.7 Rising product definition

In this short subsection, we will aim to find an expression for negative double factorials, as described by a rising product.

In Part 1, the rising product definition for negative values of the Roman factorial was given by:

$$n!_1 = n \cdot \prod_{k=1}^{-n} \frac{1}{-k}\,, \quad n \in \mathbb{Z}^-. \qquad \text{(Eq. 3.7.1)}$$

Here are some examples:

$$(-4)!_1 = (-4) \cdot \frac{1}{-1} \cdot \frac{1}{-2} \cdot \frac{1}{-3} \cdot \frac{1}{-4} = -\frac{1}{6}\,,$$
$$(-5)!_1 = (-5) \cdot \frac{1}{-1} \cdot \frac{1}{-2} \cdot \frac{1}{-3} \cdot \frac{1}{-4} \cdot \frac{1}{-5} = \frac{1}{24}\,.$$

We now need to adapt this definition for $n!_2$. To start, we modify the term inside the product by multiplying k by 2 and replacing the upper limit $-n$ with $\lceil \frac{-n}{2} \rceil$, similar to our previous approach:

$$n!_2 = n \cdot \prod_{k=1}^{\lceil \frac{-n}{2} \rceil} \frac{1}{-2k}\,, \quad n \in \mathbb{Z}^-. \qquad \text{(Eq. 3.7.2)}$$

This formula indeed works well for even n:

$$(-8)!_2 = (-8) \cdot \prod_{k=1}^{\lceil \frac{8}{2} \rceil} \frac{1}{-2k} = (-8) \cdot \prod_{k=1}^{4} \frac{1}{-2k} =$$
$$= (-8) \cdot \frac{1}{-2} \cdot \frac{1}{-4} \cdot \frac{1}{-6} \cdot \frac{1}{-8} = -\frac{1}{48}\,.$$

However, for odd n, the formula starts with $\frac{1}{-2}$ instead of $\frac{1}{-1}$:

$$(-7)!_2 = (-7) \cdot \prod_{k=1}^{\lceil \frac{7}{2} \rceil} \frac{1}{-2k} = (-7) \cdot \prod_{k=1}^{4} \frac{1}{-2k} =$$
$$= (-7) \cdot \frac{1}{-2} \cdot \frac{1}{-4} \cdot \frac{1}{-6} \cdot \frac{1}{-8} = -\frac{7}{384}\,.$$

In Subsection 2.4, we used modular arithmetic to remedy this. There, we adjusted k by subtracting $n \bmod 2$, which handled odd numbers correctly.

We remind that $n \bmod 2$ exhibits this behavior:

$$n \bmod 2 = \begin{cases} 0 & , \ n \text{ is even} \\ 1 & , \ n \text{ is odd}\,. \end{cases} \qquad \text{(Eq. 2.4.4)}$$

Here, we need to add $n \bmod 2$ to $-2k$ instead of subtracting it, so that the product starts with $\frac{1}{-1}$ as required. This gives us:

$$n!_2 = n \cdot \prod_{k=1}^{\lceil \frac{-n}{2} \rceil} \frac{1}{-2k + n \bmod 2}\,, \quad n \in \mathbb{Z}^-. \qquad \text{(Eq. 3.7.3)}$$

This adjustment ensures that for odd n, the $\prod$-product starts correctly, while nothing changes for even $n!_2$. Thus, Eq. 3.7.3 accurately describes negative double factorials as a rising product.

3.8 Synopsis

To summarize, this section introduced the concept of negative double factorials by developing 4 definitions to describe them. The recursive definition uses previous values of $n!_2$:

$$n!_2 = \begin{cases} n(n-2)!_2 & , \ n \in \mathbb{Z}^+ \\ \dfrac{(n+2)!_2}{n+2} & , \ n \in \mathbb{Z}^- \setminus \{-1, -2\}\,, \end{cases} \qquad \text{(Eq. 3.5.4)}$$

where

$$0!_2 = (-1)!_2 = (-2)!_2 = 1\,. \qquad \text{(Eq. 3.5.5)}$$

Meanwhile the Roman-like definition looks very similar to the original definition of Roman factorial:

$$\lfloor n \rceil !_2 = \begin{cases} n!_{(2)} & , \ n \in \mathbb{Z}_0^+ \\ \dfrac{(-1)^{\lceil \frac{n+1}{2} \rceil}}{(-n-2)!_{(2)}} & , \ n \in \mathbb{Z}^-, \end{cases} \qquad \text{(Eq. 3.4.8)}$$

where

$$n!_{(2)} = n(n-2)!_{(2)}\,, \quad 0!_{(2)} = (-1)!_{(2)} = 1\,, \quad n \in \mathbb{Z}^+. \qquad \text{(Eq. 3.4.3)}$$

Lastly, we described negative double factorials as a falling and as a rising product:

$n!_2$	$n \in \mathbb{Z}^-$
Falling product	$n \cdot \prod_{k=0}^{\lceil \frac{-n}{2} \rceil - 1} \frac{1}{n+2k}$
Rising product	$n \cdot \prod_{k=1}^{\lceil \frac{-n}{2} \rceil} \frac{1}{-2k + n \bmod 2}$

Tbl. 3.8.1: Double factorial expressed as a rising or falling product for negative integers

In the next section, we will integrate these definitions with those for positive integers as outlined in Section 2, further generalizing the concept as we did in Part 1.

4 Double factorial generalizations

4.1 Introduction

In this section, we will consolidate the piece-wise definitions of the double factorial that were developed in previous sections. The process we will follow is similar to the one in Part 1, with only a few modifications.

We will start by generalizing the recursive definition, followed by the Roman-like definition. After that, we will unify the two $\prod$-product definitions, and finally, we will summarize the results in a table.

To simplify the notation in this page, we will use the abbreviation $\mathbb{Z}^-_{\text{-1, -2}}$ to represent the domain of $\mathbb{Z}^- \setminus \{-1,\,-2\}$, meaning all negative integers except -1 and -2. This notation is temporary and it's only used for better readability.

4.2 Recursive definition step 1: $\theta(n)$

In this subsection, we will unify the two cases of the recursive definition of the double factorial. This generalization closely follows the analogous process in Part 1, but we will outline the steps again for clarity.

Let's start by recalling the piece-wise definition:

$$n!_2 = \begin{cases} n(n-2)!_2 & , \; n \in \mathbb{Z}^+ \\ \dfrac{(n+2)!_2}{n+2} & , \; n \in \mathbb{Z}^-_{\text{-1, -2}} \, , \end{cases} \qquad \text{(Eq. 3.5.4)}$$

where

$$0!_2 = (-1)!_2 = (-2)!_2 = 1 \, . \qquad \text{(Eq. 3.5.5)}$$

This definition is similar to the recursive definition of the Roman factorial (Eq. 1.5.4), with the key difference being the presence of the number 2 instead of 1. This minor discrepancy does not affect the generalization process, but we will detail it here.

We can begin by rewriting the cases as follows:

$$\begin{cases} n \cdot (n-2)!_2 \\ \dfrac{1}{n+2} \cdot (n+2)!_2 \end{cases} = \begin{cases} n^{+1} \cdot \big(n-(+2)\big)!_2 \\ (n+2)^{-1} \cdot \big(n-(-2)\big)!_2 \end{cases}$$

Here, we can see that the *F.F.* $\theta(n)$ fits perfectly, as it is defined by:

$$\theta(n) = \frac{\delta(n)}{|\delta(n)|} = \begin{cases} 1 & , \; n \in \mathbb{Z}^+ \\ -1 & , \; n \in \mathbb{Z}^-_{\text{-1, -2}} \, . \end{cases} \qquad \text{(Eq. 1.4.2)}$$

Although $\theta(n)$ is defined for all real numbers, here it specifically applies to the domain of the recursive definition being generalized, in order to fit it best.

By incorporating $\theta(n)$ into the definition, we obtain:

$$n!_2 = \begin{cases} n^{\theta(n)}\big(n-2\,\theta(n)\big)!_2 & , \; n \in \mathbb{Z}^+ \\ (n+2)^{\theta(n)}\big(n-2\,\theta(n)\big)!_2 & , \; n \in \mathbb{Z}^-_{\text{-1, -2}} \, . \end{cases} \qquad \text{(Eq. 4.2.1)}$$

4.3 Recursive definition step 2: $\xi'(n)$

Now, the only difference between the two cases of the recursive definition lies in the base term: n for positive integers and $(n+2)$ for $\mathbb{Z}^-_{\text{-1, -2}}$.

We express this difference as follows:

$$n!! = \begin{cases} (n+0)^{\theta(n)}\big(n-2\,\theta(n)\big)!_2 & , \; n \in \mathbb{Z}^+ \\ (n+2)^{\theta(n)}\big(n-2\,\theta(n)\big)!_2 & , \; n \in \mathbb{Z}^-_{\text{-1, -2}} \, . \end{cases} \qquad \text{(Eq. 4.3.1)}$$

This term, which can be either 0 or 2, is described by the *foundational function* $\xi'(n)$, or more precisely $2\,\xi'(n)$, as shown below:

$$2\,\xi'(n) = 1-\theta(n) = \begin{cases} 0 & , \; n \in \mathbb{Z}^+ \\ 2 & , \; n \in \mathbb{Z}^-_{\text{-1, -2}} \, . \end{cases} \qquad \text{(Eq. 1.4.4)}$$

By incorporating this function into Eq. 4.2.1, we get the same expression for both cases:

$$\begin{cases} \big(n+2\,\xi'(n)\big)^{\theta(n)}\big(n-2\,\theta(n)\big)!_2 & , \; n \in \mathbb{Z}^+ \\ \big(n+2\,\xi'(n)\big)^{\theta(n)}\big(n-2\,\theta(n)\big)!_2 & , \; n \in \mathbb{Z}^-_{\text{-1, -2}} \, . \end{cases} \qquad \text{(Eq. 4.3.2)}$$

As a result, these two cases are equivalent and can be unified into a single expression, which is the recursive definition of the double factorial[11]:

$$n!_2 = \big(n+2\,\xi'(n)\big)^{\theta(n)}\big(n-2\,\theta(n)\big)!_2 \, , \quad n \in \mathbb{D}_1 \, , \qquad \text{(Eq. 4.3.3)}$$

where

$$\mathbb{D}_1 = \{n \in \mathbb{Z} \,|\, n \neq 0,\, -1,\, -2\} \, , \qquad \text{(Eq. 4.3.4)}$$

with

$$0!_2 = (-1)!_2 = (-2)!_2 = 1 \, . \qquad \text{(Eq. 4.3.5)}$$

4.4 Roman-like definition step 1: $\eta(n, 2)$

In Subsections 4.4 to 4.6 we will unite the cases of the piece-wise definition of $n!_2$, an expression akin to the Roman factorial. Insights from Part 1 provide a clear rationale for the steps that will be followed here, and the *generalization process* is quite similar.

The piece-wise recursive definition of the double factorial is given by:

$$\lfloor n \rceil!_2 = \begin{cases} n!_{(2)} & , \; n \in \mathbb{Z}^+_0 \\ \dfrac{(-1)^{\left\lceil \frac{n+1}{2} \right\rceil}}{(-n-2)!_{(2)}} & , \; n \in \mathbb{Z}^-, \end{cases} \qquad \text{(Eq. 3.4.8)}$$

where

$$n!_{(2)} = n(n-2)!_{(2)} \, , \;\; 0!_{(2)} = (-1)!_{(2)} = 1 \, , \;\; n \in \mathbb{Z}^+ . \qquad \text{(Eq. 3.4.3)}$$

In Part 1, the first step involved introducing the function $\eta(n)$ to handle the numerator of the fraction

[11] The set-builder notation used for this definition is explained in Addendum 10.1.

in the second case of Eq. 1.5.1. since this numerator differs in the current context, we need to modify $\eta(n)$ accordingly.

We know that $\eta(n)$ is defined as:

$$\eta(n) = \theta(n)^{-n-1} = \begin{cases} 1 & , \ n \in \mathbb{Z}_0^+ \\ (-1)^{-n-1} & , \ n \in \mathbb{Z}^- . \end{cases} \quad \text{(Eq. 1.4.5)}$$

Instead of modifying $\eta(n)$ for the purposes of $n!_2$, a better approach is to create an entirely new function that takes two variables. Let's define $\eta(n,2)$ as a two-variable function of the form:

$$\eta(n,2) = \theta(n)^{\left\lceil \frac{n+1}{2} \right\rceil} = \begin{cases} 1 & , \ n \in \mathbb{Z}_0^+ \\ (-1)^{\left\lceil \frac{n+1}{2} \right\rceil} & , \ n \in \mathbb{Z}^- . \end{cases} \quad \text{(Eq. 4.4.1)}$$

In this context, the variable 2 represents the order of the factorial. As we will see in Section 5, the traditional factorial is of order 1, while the double factorial is of order 2. By redefining $\eta(n)$ as a two-variable function, we prepare for an easier generalization of the Roman-like definition of the multifactorial later on.

We can now incorporate $\eta(n,2)$ into Eq. 3.4.8 as follows:

$$\begin{cases} 1 \cdot n!_{(2)} \\ \dfrac{(-1)^{\left\lceil \frac{n+1}{2} \right\rceil}}{(-n-2)!_{(2)}} \end{cases} = \begin{cases} \eta(n,2) \cdot n!_{(2)} & , \ n \in \mathbb{Z}_0^+ \\ \dfrac{\eta(n,2)}{(-n-2)!_{(2)}} & , \ n \in \mathbb{Z}^- . \end{cases}$$

Thus, we have completed the first step in our quest to generalize the Roman-like definition of the factorial of order 2. The generalized definition is:

$$\lfloor n \rceil !_2 = \begin{cases} \eta(n,2) \cdot n!_{(2)} & , \ n \in \mathbb{Z}_0^+ \\ \dfrac{\eta(n,2)}{(-n-2)!_{(2)}} & , \ n \in \mathbb{Z}^- . \end{cases} \quad \text{(Eq. 4.4.2)}$$

where

$$n!_{(2)} = n(n-2)!_{(2)} \, , \ \ 0!_{(2)} = (-1)!_{(2)} = 1 \, , \ \ n \in \mathbb{Z}^+ . \quad \text{(Eq. 3.4.3)}$$

4.5 Roman-like definition step 2: $\theta(n)$

In this step, we will emphasize the exponent of the double factorial in Eq. 4.4.2, leading to:

$$\lfloor n \rceil !_2 = \begin{cases} \eta(n,2) \cdot \left(n!_{(2)}\right)^1 & , \ n \in \mathbb{Z}_0^+ \\ \eta(n,2) \cdot \left[(-n-2)!_{(2)}\right]^{-1} & , \ n \in \mathbb{Z}^- . \end{cases} \quad \text{(Eq. 4.5.1)}$$

This reveals another instance of $\theta(n)$, as given by

$$\theta(n) = \frac{\delta(n)}{|\delta(n)|} = \begin{cases} 1 & , \ n \in \mathbb{Z}^+ \\ -1 & , \ n \in \mathbb{Z}^- . \end{cases} \quad \text{(Eq. 1.4.2)}$$

We can now substitute the highlighted exponents with $\theta(n)$, resulting in:

$$\lfloor n \rceil !_2 = \begin{cases} \eta(n,2) \cdot \left(n!_{(2)}\right)^{\theta(n)} & , \ n \in \mathbb{Z}_0^+ \\ \eta(n,2) \cdot \left[(-n-2)!_{(2)}\right]^{\theta(n)} & , \ n \in \mathbb{Z}^- . \end{cases} \quad \text{(Eq. 4.5.2)}$$

where

$$n!_{(2)} = n(n-2)!_{(2)} \, , \ \ 0!_{(2)} = (-1)!_{(2)} = 1 \, , \ \ n \in \mathbb{Z}^+ . \quad \text{(Eq. 3.4.3)}$$

Before proceeding further, we can make two improvements for clarity. First, the parentheses around the double factorial can be removed to simplify the notation. The definition now becomes:

$$\lfloor n \rceil !_2 = \begin{cases} \eta(n,2) \cdot n!_{(2)}^{\theta(n)} & , \ n \in \mathbb{Z}_0^+ \\ \eta(n,2) \cdot (-n-2)!_{(2)}^{\theta(n)} & , \ n \in \mathbb{Z}^- . \end{cases} \quad \text{(Eq. 4.5.3)}$$

We can further simplify the expression by replacing $-n$ in the second case with $|n|$, and similarly adjust the first case:

$$\lfloor n \rceil !_2 = \begin{cases} \eta(n,2) \cdot |n|!_{(2)}^{\theta(n)} & , \ n \in \mathbb{Z}_0^+ \\ \eta(n,2) \cdot \left(|n|-2\right)!_{(2)}^{\theta(n)} & , \ n \in \mathbb{Z}^- . \end{cases} \quad \text{(Eq. 4.5.4)}$$

This transformation is possible because of the behavior of the absolute value of n, defined as:

$$|n| = \begin{cases} n & , \ n \geq 0 \\ -n & , \ n < 0 \, . \end{cases} \quad \text{(Eq. 4.5.5)}$$

4.6 Roman-like definition step 3: $\xi'(n)$

In the third and final step of this generalization, we address the subtraction of 2 from $|n|$, which is absent in the first case but present in the second. This discrepancy is highlighted below:

$$\lfloor n \rceil !_2 = \begin{cases} \eta(n,2) \cdot \left(|n|-0\right)!_{(2)}^{\theta(n)} & , \ n \in \mathbb{Z}_0^+ \\ \eta(n,2) \cdot \left(|n|-2\right)!_{(2)}^{\theta(n)} & , \ n \in \mathbb{Z}^- . \end{cases} \quad \text{(Eq. 4.6.1)}$$

This pattern is reminiscent of the *F.F.* $\xi'(n)$:

$$\xi'(n) = \frac{1-\theta(n)}{2} = \begin{cases} 0 & , \ n \in \mathbb{Z}_0^+ \\ 1 & , \ n \in \mathbb{Z}^- . \end{cases} \quad \text{(Eq. 1.4.4)}$$

Here, $\xi'(n)$ has the output pattern $[0,1]$ but to match the required subtraction in the second case of our expression, we need the outputs $[0,2]$. This can be achieved by multiplying $\xi'(n)$ by 2, and this expression can be directly replaced into Eq. 4.6.1:

$$\lfloor n \rceil !_2 = \begin{cases} \eta(n,2) \cdot \left(|n|-2\,\xi'(n)\right)!_{(2)}^{\theta(n)} & , \ n \in \mathbb{Z}_0^+ \\ \eta(n,2) \cdot \left(|n|-2\,\xi'(n)\right)!_{(2)}^{\theta(n)} & , \ n \in \mathbb{Z}^- . \end{cases} \quad \text{(Eq. 4.6.2)}$$

This completes the *generalization process*, resulting in a unified Roman-like expression for the double factorial:

$$\lfloor n \rceil !_2 = \eta(n,2) \cdot \left(|n|-2\,\xi'(n)\right)!_{(2)}^{\theta(n)} \, , \quad n \in \mathbb{Z} \, , \quad \text{(Eq. 4.6.3)}$$

where

$$n!_{(2)} = n(n-2)!_{(2)} \, , \ \ 0!_{(2)} = (-1)!_{(2)} = 1 \, , \ \ n \in \mathbb{Z}^+ . \quad \text{(Eq. 3.4.3)}$$

4.7 $\prod$-product definitions step 1: $|n|$

In this set of subsections, we will generalize the non-recursive, $\prod$-product definitions of the factorial of order 2 shown earlier.

To begin, let's present these two definitions in tabular form. The first table contains the falling product definitions for both positive and negative integers (Subsections 2.3 and 3.6), while the second table lists the two cases of the rising product that were found in Subsections 2.4 and 3.7:

$n!_2$	Falling product
$n \in \mathbb{Z}^+$	$\displaystyle\prod_{k=0}^{\lceil \frac{n}{2} \rceil - 1} (n-2k)$
$n \in \mathbb{Z}^-$	$\displaystyle n \cdot \prod_{k=0}^{\lceil \frac{-n}{2} \rceil - 1} \frac{1}{n+2k}$

Tbl. 4.7.1: $n!_2$ as a falling product

$n!_2$	Rising product
$n \in \mathbb{Z}^+$	$\displaystyle\prod_{k=1}^{\lceil \frac{n}{2} \rceil} (2k - n \bmod 2)$
$n \in \mathbb{Z}^-$	$\displaystyle n \cdot \prod_{k=1}^{\lceil \frac{-n}{2} \rceil} \frac{1}{-2k + n \bmod 2}$

Tbl. 4.7.2: $n!_2$ as a rising product

Notice that $n = 0$ is missing from both cases in each $\prod$-product. Here, we take advantage of the concept of the empty product, and include $n = 0$ in the first case of both $\prod$-products.

To confirm the validity of this assessment, let's perform the calculations:

$$n!_2 = \prod_{k=0}^{\lceil \frac{0}{2} \rceil - 1} (0 - 2k) = \prod_{k=0}^{-1} (-2k) = 1 \,,$$

$$n!_2 = \prod_{k=1}^{\lceil \frac{0}{2} \rceil} (2k - 0 \bmod 2) = \prod_{k=1}^{0} 2k = 1 \,.$$

Recall that the empty product arises when the upper limit of a $\prod$-product is less than its lower limit, resulting in a product value of 1 regardless of the expression for k. For more information about the empty product and $\prod$-products in general, check out Addendum 10.4.

Continuing with the $\prod$-product definitions, note that the upper limits of the products are expressed as $\lceil \frac{n}{2} \rceil$ for $n \in \mathbb{Z}_0^+$ and as $\lceil \frac{-n}{2} \rceil$ for $n \in \mathbb{Z}^-$. The negative sign ensures that the upper limits remain positive when n is negative.

We can eliminate the negative sign by replacing n with $|n|$. This leads to the following definitions:

$n!_2$	Falling product
$n \in \mathbb{Z}_0^+$	$\displaystyle\prod_{k=0}^{\lceil \frac{\lvert n \rvert}{2} \rceil - 1} (n-2k)$
$n \in \mathbb{Z}^-$	$\displaystyle n \cdot \prod_{k=0}^{\lceil \frac{\lvert n \rvert}{2} \rceil - 1} \frac{1}{n+2k}$

Tbl. 4.7.3: $n!_2$ as a falling product (generalization step 1)

$n!_2$	Rising product
$n \in \mathbb{Z}_0^+$	$\displaystyle\prod_{k=1}^{\lceil \frac{\lvert n \rvert}{2} \rceil} (2k - n \bmod 2)$
$n \in \mathbb{Z}^-$	$\displaystyle n \cdot \prod_{k=1}^{\lceil \frac{\lvert n \rvert}{2} \rceil} \frac{1}{-2k + n \bmod 2}$

Tbl. 4.7.4: $n!_2$ as a rising product (generalization step 1)

This step in the *generalization process* is identical to the corresponding one in Part 1, repeated here in detail to show every part of the process without skipping any steps, even if they seem repetitive.

4.8 $\prod$-product definitions step 2: $\Phi(n)$

We will now proceed to incorporate the *F.F.* $\Phi(n)$ into our definitions. It was defined in Part 1 as:

$$\Phi(n) = \big(n + \Theta(n)\big)^{\xi'(n)} = \begin{cases} 1\,, & n \in \mathbb{Z}_0^+ \\ n\,, & n \in \mathbb{Z}^- . \end{cases}$$

(Eq. 1.4.10)

Since we can state that the positive cases in the $\prod$-product definitions are multiplied by 1, we can seamlessly substitute $\Phi(n)$ there:

$n!_2$	Falling product
$n \in \mathbb{Z}_0^+$	$\displaystyle \Phi(n) \cdot \prod_{k=0}^{\lceil \frac{\lvert n \rvert}{2} \rceil - 1} (n-2k)$
$n \in \mathbb{Z}^-$	$\displaystyle \Phi(n) \cdot \prod_{k=0}^{\lceil \frac{\lvert n \rvert}{2} \rceil - 1} \frac{1}{n+2k}$

Tbl. 4.8.1: $n!_2$ as a falling product (generalization step 2)

$n!_2$	Rising product
$n \in \mathbb{Z}_0^+$	$\Phi(n) \cdot \prod_{k=1}^{\lceil \frac{\lvert n \rvert}{2} \rceil} (2k - n \bmod 2)$
$n \in \mathbb{Z}^-$	$\Phi(n) \cdot \prod_{k=1}^{\lceil \frac{\lvert n \rvert}{2} \rceil} \frac{1}{-2k + n \bmod 2}$

Tbl. 4.8.2: $n!_2$ as a rising product (generalization step 2)

With this, we have successfully completed step 2 of the *generalization process*. The next and final step will allow us to summarize our results and bring this section to a conclusion.

4.9 $\prod$-product definitions step 3: $\theta(n)$

In this final step, we analyze the differences in the index terms for the $\prod$-product definitions and consolidate them. The index expressions for both the falling and rising products are summarized in the following table:

Indices	Falling product	Rising product
$n \in \mathbb{Z}_0^+$	$n - 2k$	$(2k - n \bmod 2)$
$n \in \mathbb{Z}^-$	$\frac{1}{n + 2k}$	$\frac{1}{-2k + n \bmod 2}$

Tbl. 4.9.1: Index terms in the $\prod$-product definitions of the double factorial

The term "indices" here refers to the index terms in the $\prod$-product, and it will be used as an abbreviation in this context.

Notably, $\theta(n)$ appears prominently in each case. The positive cases for $n \in \mathbb{Z}_0^+$ have expressions that invert when $n \in \mathbb{Z}^-$. This is equivalent to raising these expressions to the power of $\theta(n)$:

Indices	Falling product	Rising product
$n \in \mathbb{Z}_0^+$	$(n - 2k)^{\theta(n)}$	$(2k - n \bmod 2)^{\theta(n)}$
$n \in \mathbb{Z}^-$	$(n + 2k)^{\theta(n)}$	$(-2k + n \bmod 2)^{\theta(n)}$

Tbl. 4.9.2: Index terms in the $\prod$-product definitions of the double factorial, with highlighted exponents

There are still some occurrences of this function that need to be addressed. For the falling product, the term $2k$ changes sign when n does, which allows us to rewrite the expression as $\big(n - 2k\,\theta(n)\big)$. In the rising product, the entire expression changes sign. This can be represented as $\theta(n) \cdot (2k - n \bmod 2)$.

Incorporating these changes gives us the following table for the expressions inside the products:

Indices	$n \in \mathbb{Z}$
Falling product	$\big(n - 2k\,\theta(n)\big)^{\theta(n)}$
Rising product	$\theta(n) \cdot (2k - n \bmod 2)^{\theta(n)}$

Tbl. 4.9.3: Index terms in the $\prod$-product definitions of the double factorial, unified

With the index terms now unified, the $\prod$-product definitions themselves can be consolidated.

The final expressions are presented below, where Eq. 4.9.1 is the generalized falling product definition of $n!_2$ and Eq. 4.9.2 its rising product formulation:

$$n!_2 = \Phi(n) \cdot \prod_{k=0}^{\lceil \frac{\lvert n \rvert}{2} \rceil - 1} \big(n - 2k\,\theta(n)\big)^{\theta(n)}, \quad n \in \mathbb{Z}, \qquad \text{(Eq. 4.9.1)}$$

$$n!_2 = \Phi(n) \cdot \prod_{k=1}^{\lceil \frac{\lvert n \rvert}{2} \rceil} \big(\theta(n) \cdot (2k - n \bmod 2)\big)^{\theta(n)}, \; n \in \mathbb{Z}. \qquad \text{(Eq. 4.9.2)}$$

4.10 Synopsis

In Section 4, we explored various generalizations of the double factorial. The generalized recursive definition for $n!_2$ is:

$$n!_2 = \big(n + 2\,\xi'(n)\big)^{\theta(n)} \big(n - 2\,\theta(n)\big)!_2\,, \quad n \in \mathbb{D}_1\,, \qquad \text{(Eq. 4.3.3)}$$

where

$$\mathbb{D}_1 = \{n \in \mathbb{Z} \,|\, n \neq 0,\, -1,\, -2\}\,, \qquad \text{(Eq. 4.3.4)}$$

with

$$0!_2 = (-1)!_2 = (-2)!_2 = 1\,. \qquad \text{(Eq. 4.3.5)}$$

Next, the generalized Roman-like definition is:

$$\lfloor n \rceil !_2 = \eta(n, 2) \cdot \big(\lvert n \rvert - 2\,\xi'(n)\big)!_{(2)}^{\theta(n)}\,, \quad n \in \mathbb{Z}\,, \qquad \text{(Eq. 4.6.3)}$$

where

$$n!_{(2)} = n(n-2)!_{(2)}\,, \quad 0!_{(2)} = (-1)!_{(2)} = 1\,, \quad n \in \mathbb{Z}^+. \qquad \text{(Eq. 3.4.3)}$$

Finally, the generalized $\prod$-product definitions of $n!_2$ as a falling or as a rising product are defined as:

$$n!_2 = \Phi(n) \cdot \prod_{k=0}^{\lceil \frac{\lvert n \rvert}{2} \rceil - 1} \big(n - 2k\,\theta(n)\big)^{\theta(n)}, \quad n \in \mathbb{Z}, \qquad \text{(Eq. 4.9.1)}$$

$$n!_2 = \Phi(n) \cdot \prod_{k=1}^{\lceil \frac{\lvert n \rvert}{2} \rceil} \big(\theta(n) \cdot (2k - n \bmod 2)\big)^{\theta(n)}, \; n \in \mathbb{Z}. \qquad \text{(Eq. 4.9.2)}$$

With these generalized definitions established, we are now prepared to explore factorials of higher order, known as multifactorials.

5 Multifactorial

5.1 Introduction

The multifactorial, denoted as $n!_{(m)}$, is a generalization of the traditional factorial, where the traditional factorial is of order 1, and the double factorial is of order 2. The multifactorial extends this concept to any positive integer order m, where m stands for the order of the factorial. For the purposes of this paper it will always be a natural number ($m \in \mathbb{N}$).

In essence, $n!_{(m)}$ is the product of every m-th integer, starting from n and descending until the lowest positive integer. Examples for various values of m are as follows:

$$8!_{(3)} = 8 \cdot 5 \cdot 2 = 80\,,$$
$$11!_{(4)} = 11 \cdot 7 \cdot 3 = 231\,,$$
$$16!_{(5)} = 16 \cdot 11 \cdot 6 \cdot 1 = 1056\,.$$

This view of the multifactorial as a falling product is a useful starting point, but in Subsection 5.4 we will define it as a rising product too.

Below, we provide a table listing multifactorial values up to $m = 4$:

n	0	1	2	3	4	5	6	7
$n!_{(1)}$	1	1	2	6	24	120	720	5040
$n!_{(2)}$	1	1	2	3	8	15	48	105
$n!_{(3)}$	1	1	2	3	4	10	18	28
$n!_{(4)}$	1	1	2	3	4	5	12	21

Tbl. 5.1.1: Multifactorials

The following figure illustrates[12] these multifactorials in a nice way:

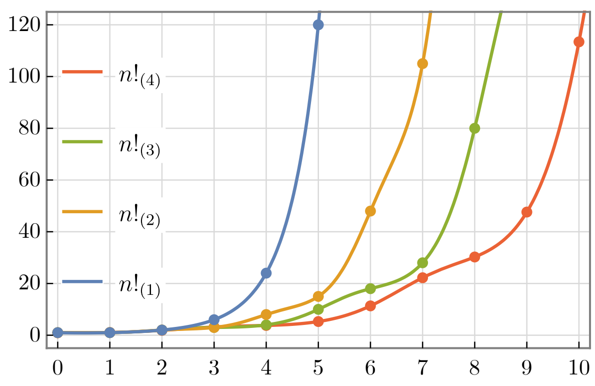

Fig. 5.1.1: Multifactorial

As the order m increases, the multifactorial grows more slowly. This behavior is expected since increasing m reduces the number of terms in the product, which contains every m-th integer.

In the remainder of this introduction, we will derive the recursive properties of the multifactorial from the cases of $m = 1$ and $m = 2$.

Firstly, let's recall the recursive properties of the traditional and double factorials, which were stated in Subsection 2.2:

$$n!_{(1)} = n(n-1)!_{(1)}\,, \quad n \in \mathbb{Z}^+, \qquad \text{(Eq. 2.2.1)}$$
$$n!_{(2)} = n(n-2)!_{(2)}\,, \quad n \in \mathbb{Z}^+ \setminus \{1\}\,. \qquad \text{(Eq. 2.2.2)}$$

From these relations, we derive the general recursive definition for the multifactorial:

$$n!_{(m)} = n(n-m)!_{(m)}\,, \qquad \text{(Eq. 5.1.1)}$$

where n must be in an appropriate domain for the equation to hold.

Since multifactorials are defined for $n \in \mathbb{Z}_0^+$, the smallest possible multifactorial is $0!_{(m)} = 1$. The recursive property we are looking for relates $n!_{(m)}$ with $(n-m)!_{(m)}$, which is itself a smaller factorial.

Thus, the domain of Eq. 5.1.1 is all positive integers down to $n = m$, as otherwise $(n-m)!_{(m)}$ would be a negative multifactorial, which is not currently defined.

Therefore, the recursive property of the multifactorial now becomes:

$$n!_{(m)} = n(n-m)!_{(m)}\,, \quad \{n \in \mathbb{Z} \,|\, n \geq m\}\,. \qquad \text{(Eq. 5.1.2)}$$

where the notation[13] $\{n \in \mathbb{Z} \,|\, n \geq m\}$ means the set of all integers greater or equal to m.

There is an alternative form of Eq. 5.1.2 that was examined for the traditional and the double factorials. If we substitute n with $n + m$ and solve for $n!_{(m)}$, we obtain another recursive property:

$$n!_{(m)} = \frac{(n+m)!_{(m)}}{n+m}\,, \quad n \in \mathbb{Z}_0^+. \qquad \text{(Eq. 5.1.3)}$$

This equation relates a multifactorial with a larger one, making the domain $\mathbb{Z}_0^+$ appropriate.

In the next subsection, we will expand the domain of these recursive properties to include some negative integers. Although multifactorials are primarily defined for positive integers, extending them to include negative values simplifies the recursive definition.

To conclude this introduction, we've presented the concept of multifactorials and defined two recursive properties. In the following sections, we will derive a more comprehensive recursive definition and explore the $\prod$-product formulations, describing $n!_{(m)}$ as a falling/rising product.

The structure of this section mirrors Section 2 of this paper, and a similar design will be followed in subsequent sections.

[12] Continuations of $n!_{(m)}$ into non-integers will be discussed in the next part of this study. Fig. 5.1.1 was made using a formula found in Addendum 10.3.

[13] For more information about number sets and the set-builder notation, check Addendum 10.1.

5.2 Recursive definition

As mentioned in Subsection 3.5, a recursive definition is well-defined when it includes a recursive relationship along with a set of initial conditions, or seeds. In this subsection, we will derive the recursive definition for the multifactorial by generalizing from the cases of $m = 1$ and $m = 2$.

Let's start by recalling the recursive definitions of the traditional and double factorials:

$$n!_{(1)} = n(n-1)!_{(1)}\,, \quad 0!_{(1)} = 1\,, \quad n \in \mathbb{Z}^+, \qquad \text{(Eq. 1.5.2)}$$

$$n!_{(2)} = n(n-2)!_{(2)}\,, \quad 0!_{(2)} = 1!_{(2)} = 1\,, \quad n \in \mathbb{Z}^+ \setminus \{1\}\,. \qquad \text{(Eq. 2.2.3)}$$

It is evident that Eq. 5.1.2 requires a set of seeds to fully define it. Let's explore this further.

The traditional factorial needs only one starting value to cover every integer ($0! = 1$), as the recursive relationship naturally links consecutive integers. In contrast, the double factorial requires two seeds to achieve the same thing ($0!! = 1!! = 1$), since it involves the product of every other integer. A single seed would only cover half of all $n \in \mathbb{Z}_0^+$.

By analogy, the multifactorial would require m seeds to encompass every possible integer. Its product consists of every m-th integer, so it makes sense that m starting values are needed to define it recursively for any possible $n \in \mathbb{Z}_0^+$.

Typically, we would choose the first m integers as seeds. However, this approach encounters complications. Consider the triple factorial $n!_{(3)}$ as an example:

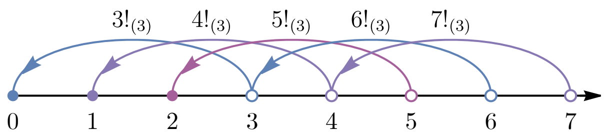

Fig. 5.2.1: Triple factorial recursiveness

In this figure, we see that $n!_{(3)}$ for non-negative integers requires three seeds to define the triple factorial recursively. The seeds are $0!_{(3)} = 1!_{(3)} = 1$ and $2!_{(3)} = 2$. While the first two seeds are set to 1, the third seed must be set to 2; otherwise, the recursive relationship would yield incorrect results (for instance, $8!_{(3)}$ would not equal the product $8 \cdot 5 \cdot 2$ anymore).

If we had considered $n!_{(4)}$ instead, we would need a fourth seed: $3!_{(4)} = 3$. Generally, every seed except $0!_{(m)}$ must be set to $n!_{(m)} = n$, where the seeds belong in $\{n \in \mathbb{Z} \,|\, 0 \leq n < m\}$.

We will adhere to this approach for now, even though the resulting definition might not be as intuitive or useful as an alternative one that will be presented afterwards.

The seeds from 0 to $m-1$ can all be systemically set using the *F.F.* $\Psi(n)$, which is defined as:

$$\Phi(n) = \big(n + \Theta(n)\big)^{\xi'(n)} = \begin{cases} 1\,, & n = 0 \\ n\,, & n \in \mathbb{Z}^+. \end{cases} \qquad \text{(Eq. 1.4.10)}$$

This function allows us to set $n!_{(m)} = \Psi(n)$ for $\{n \in \mathbb{Z} \,|\, 0 \leq n < m\}$ as the seeds for the multifactorial, leading to the following form of its recursive definition:

$$n!_{(m)} = n(n-m)!_{(m)}\,, \quad \{n \in \mathbb{Z} \,|\, n \geq m\}\,, \qquad \text{(Eq. 5.2.1)}$$

where

$$n!_{(m)} = \Psi(n)\,, \quad \{n \in \mathbb{Z} \,|\, 0 \leq n < m\}\,. \qquad \text{(Eq. 5.2.2)}$$

This expression, however, is more complicated than necessary, and there is a way to avoid using a *F.F.* in the recursive definition of the multifactorial. The seeds can be chosen more straightforwardly, leading to a simpler and more elegant definition.

To simplify the recursive definition, let's consider extending the domain of $n!_{(m)}$ is defined in, by applying a method similar to how double factorials were extended to negative odd integers using the recursive relationship from Eq. 3.2.1.

We will focus initially on the case of the triple factorial. The recursive property that is analogous to Eq. 3.2.1 for the triple factorial can be expressed as:

$$n!_{(3)} = \frac{(n+3)!_{(3)}}{n+3}\,, \quad n \in \mathbb{Z}_0^+. \qquad \text{(Eq. 5.2.3)}$$

Although negative triple factorials have not yet been explicitly defined, we can test various values beyond $\mathbb{Z}_0^+$ to verify the validity of this recursive relationship. The value $n = -3$ is of course undefined, so we are limited to the first two negative integers.

By evaluating Eq. 5.2.3 for $n = -1$ and -2, we obtain:

$$(-1)!_{(3)} = \frac{(-1+3)!_{(3)}}{-1+3} = \frac{2!_{(3)}}{2} = 1\,,$$

$$(-2)!_{(3)} = \frac{(-2+3)!_{(3)}}{-2+3} = \frac{1!_{(3)}}{1} = 1\,.$$

This approach will be applied in more detail in Subsection 6.2 for the general case of $n!_{(m)}$, but for now, it helps us in finding a simpler recursive definition for the multifactorial.

From the examples above, we see that some negative triple factorials can be computed without error using the recursive property. Accepting that $n!_{(3)}$ can be extended to include the first two negative integers in this way allows us to redefine the seeds in its recursive definition.

This concept is further clarified in the following illustration:

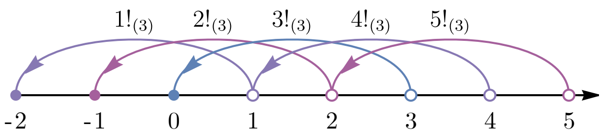

Fig. 5.2.2: Triple factorial recursiveness with a different set of starting values

As shown in Fig. 5.2.2, exactly three seeds are required to define $n!_{(3)}$ in this manner: $n = -2, -1$ and 0. Notably, these seeds all share the same value: $(-2)!_{(3)} = (-1)!_{(3)} = 0!_{(3)} = 1$.

By adopting this approach, the triple factorial can now be defined recursively as:

$$n!_{(3)} = n(n-3)!_{(3)}\,, \quad n \in \mathbb{Z}^+, \qquad \text{(Eq. 5.2.4)}$$

where

$$(-2)!_{(3)} = (-1)!_{(3)} = 0!_{(3)} = 1\,. \qquad \text{(Eq. 5.2.5)}$$

To solidify this concept, let's briefly consider another example. The recursive behavior of the quadruple factorial $n!_{(4)}$ is depicted below:

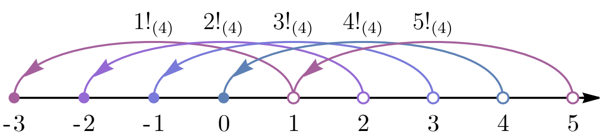

Fig. 5.2.3: Quadruple factorial recursiveness

Here, it is evident that four seeds are necessary to define the quadruple factorial recursively for all integers. The seeds $(-3)!_{(4)}$ through $0!_{(4)}$ can all be set to 1, ensuring the correct values for the quadruple factorial as seen in Tbl. 5.1.1.

In conclusion, we generalize this idea for the $n!_{(m)}$ based on our analysis of factorials for $m = 1$ to $m = 4$. For $m = 1$, only the seed $0!_{(1)} = 1$ is required, while the double factorial in Eq. 3.5.3 needs the seeds $(-1)!_{(2)} = 0!_{(2)} = 1$. Similarly, the triple factorial requires the seeds $(-2)!_{(3)} = (-1)!_{(3)} = 0!_{(3)} = 1$.

For the factorial of order 4, as seen in Fig. 5.2.3, an additional seed $(-3)!_{(4)} = 1$ is necessary. Hence, the general case $n!_{(m)}$ can be well-defined recursively by using the following starting values: $(-m+1)!_{(m)} = (-m+2)!_{(m)} = \cdots = (-1)!_{(m)} = 0!_{(m)} = 1$.

Thus, the recursive definition of the multifactorial is:

$$n!_{(m)} = n(n-m)!_{(m)}\,, \quad n \in \mathbb{Z}^+, \qquad \text{(Eq. 5.2.6)}$$

where

$$n!_{(m)} = 1\,, \quad \{n \in \mathbb{Z} \,|\, -m < n \leq 0\}\,. \qquad \text{(Eq. 5.2.7)}$$

This concludes the development of the recursive definition of the multifactorial, which is now applicable to a broader range of integers. A thorough analysis of its behavior for all negative integers will be performed in Subsection 6.2.

5.3 Falling product definition

In this subsection, we will follow a similar logic to find the falling product definition for the multifactorial. Specifically, we will recall the falling products of the traditional and double factorial, make an educated guess for the general case, and then verify its validity through specific examples.

Let's begin by recalling the falling product definition of the traditional factorial, as it was discussed in Part 1:

$$n!_{(1)} = \prod_{k=0}^{n-1} (n-k)\,, \quad n \in \mathbb{Z}^+. \qquad \text{(Eq. 2.3.1)}$$

This product iterates over the entire sequence of positive integers from n down to 1, essentially multiplying each term in the sequence until it reaches the base case. The factorial of any positive integer is thus the product of all integers less than or equal to that number.

The falling product definition of the double factorial, as formulated in Subsection 2.3, is as follows:

$$n!_{(2)} = \prod_{k=0}^{\lceil \frac{n}{2} \rceil - 1} (n-2k)\,, \quad n \in \mathbb{Z}^+. \qquad \text{(Eq. 2.3.4)}$$

Here, the sequence skips every other integer, reflecting the fact that the double factorial involves multiplying only the odd or even integers, depending on whether the initial number is odd or even. Notice the term $\lceil \frac{n}{2} \rceil$ in the upper limit of the product. This term adjusts the range of the product, ensuring it accounts only for the required factors in the double factorial.

In Eq. 2.3.4, there is a term in the upper limit of the product that does not appear in the traditional factorial: the term $\lceil \frac{n}{2} \rceil$ is just n in the other definition. However, upon closer examination, these quantities are closely related.

The double factorial is considered to be of order 2, while the traditional factorial of order 1. Since the former contains the term $\lceil \frac{n}{2} \rceil$ in its falling product definition, it follows logically that the latter should include the term $\lceil \frac{n}{1} \rceil$ in its product limit.

Let's verify this assessment for $m = 1$. Indeed, we can confirm that the claim above is true:

$$\left\lceil \frac{n}{1} \right\rceil = \lceil n \rceil = n\,, \quad n \in \mathbb{Z}^+.$$

This reasoning implies that in the falling product definition of $m = 1$ there is the term $\lceil \frac{n}{1} \rceil$, and for $m = 2$ the corresponding term is $\lceil \frac{n}{2} \rceil$. It is safe to assume that for $m = 3$, this term is just $\lceil \frac{n}{3} \rceil$.

We can now extend this logic to formulate a falling product definition for the triple factorial. By adjusting the expression of k inside the product accordingly, we obtain the following:

$$n!_{(3)} = \prod_{k=0}^{\lceil \frac{n}{3} \rceil - 1} (n-3k)\,, \quad n \in \mathbb{Z}^+. \qquad \text{(Eq. 5.3.1)}$$

To test the validity of this definition, let's compute a few examples:

$$6!_{(3)} = \prod_{k=0}^{\lceil \frac{6}{3} \rceil - 1} (6 - 3k) = \prod_{k=0}^{1} (6 - 3k) = 6 \cdot 3 = 18\,,$$

$$7!_{(3)} = \prod_{k=0}^{\lceil \frac{7}{3} \rceil - 1} (7 - 3k) = \prod_{k=0}^{2} (6 - 3k) = 7 \cdot 4 \cdot 1 = 28\,,$$

$$8!_{(3)} = \prod_{k=0}^{\lceil \frac{8}{3} \rceil - 1} (8 - 3k) = \prod_{k=0}^{2} (6 - 3k) = 8 \cdot 5 \cdot 2 = 80\,.$$

The results confirm that Eq. 5.3.1 accurately describes the triple factorial for these examples, which suggests that the proposed definition is indeed correct.

To further solidify this approach, let's extend the analysis to $m = 4$. The falling product definition in this case would be:

$$n!_{(4)} = \prod_{k=0}^{\lceil \frac{n}{4} \rceil - 1} (n - 4k)\,, \quad n \in \mathbb{Z}^+. \qquad \text{(Eq. 5.3.2)}$$

Eq. 5.3.2 produces the following factorials:

$$8!_{(4)} = \prod_{k=0}^{\lceil \frac{8}{4} \rceil - 1} (8 - 4k) = \prod_{k=0}^{1} (8 - 4k) = 8 \cdot 4 = 32\,,$$

$$9!_{(4)} = \prod_{k=0}^{\lceil \frac{9}{4} \rceil - 1} (9 - 4k) = \prod_{k=0}^{2} (9 - 4k) = 9 \cdot 5 \cdot 1 = 45\,,$$

$$10!_{(4)} = \prod_{k=0}^{\lceil \frac{10}{4} \rceil - 1} (10 - 4k) = \prod_{k=0}^{2} (10 - 4k) = 10 \cdot 6 \cdot 2\,,$$

$$11!_{(4)} = \prod_{k=0}^{\lceil \frac{11}{4} \rceil - 1} (11 - 4k) = \prod_{k=0}^{2} (11 - 4k) = 11 \cdot 7 \cdot 3\,.$$

As expected, Eq. 5.3.2 produces accurate values for the quadruple factorial, which shows that our method works well. The consistent results across different examples suggest that the falling product definition we've proposed for any multifactorial can be successfully generalized.

Thus, the general case can be written as:

$$n!_{(m)} = \prod_{k=0}^{\lceil \frac{n}{m} \rceil - 1} (n - mk)\,, \quad n \in \mathbb{Z}^+. \qquad \text{(Eq. 5.3.3)}$$

This definition is considered the falling product definition of the multifactorial. It is important to note that it is not defined for $n = 0$, but that is simply because we have not yet considered this specific case. However, extending this definition to include $n = 0$ is straightforward.

If we try to calculate $0!_{(m)}$ using Eq. 5.3.3, we get:

$$0!_{(m)} = \prod_{k=0}^{\lceil \frac{0}{m} \rceil - 1} (0 - mk) = \prod_{k=0}^{-1} (-mk) = 1\,.$$

In this scenario, the upper limit is less than the lower limit, so the resulting product is always 1, regardless of the value of $(-mk)$ since it resembles the empty product. This allows us to expand the domain of Eq. 5.3.3 from $\mathbb{Z}^+$ to $\mathbb{Z}_0^+$, thereby providing a definition that covers all non-negative integers.

In fact, we can take this one step further. In the previous subsection, we defined the multifactorial recursively for some negative integers as well. Since the upper limit of the product in Eq. 5.3.3 becomes increasingly negative as n decreases, it makes sense that the product will always equal 1 due to the empty product property. Therefore, the expanded domain will now take the form $\{n \in \mathbb{Z} \mid -m < n\}$.

In conclusion, the expanded falling product definition of the multifactorial is hereby defined as:

$$n!_{(m)} = \prod_{k=0}^{\lceil \frac{n}{m} \rceil - 1} (n - mk)\,, \quad \{n \in \mathbb{Z} \mid -m < n\}\,. \qquad \text{(Eq. 5.3.4)}$$

This definition not only encompasses the traditional and double factorials but also generalizes to multifactorials as well. With this foundation, we are now ready to explore the rising product definition of the multifactorial, which will complete our analysis in this section.

5.4 Rising product definition

We will start this subsection by revisiting the rising product definitions for the traditional and double factorials. We will use a similar method to explore the general case, following the approach outlined earlier in this section.

The definitions are as follows:

$$n!_{(1)} = \prod_{k=1}^{n} k\,, \quad n \in \mathbb{Z}^+, \qquad \text{(Eq. 2.4.1)}$$

$$n!_{(2)} = \prod_{k=1}^{\lceil \frac{n}{2} \rceil} (2k - n \bmod 2)\,, \quad n \in \mathbb{Z}^+. \qquad \text{(Eq. 2.4.5)}$$

These rising product definitions seem less straightforward compared to their counterparts in the falling product case for $m = 1$ and $m = 2$. Nonetheless, we will apply the same reasoning as before to see where it leads us.

The upper limit in the product follows the same pattern as in the falling product definitions. This consistency is expected since it reflects the number of factors in the final product, and the difference between rising and falling products lies only in the order of these factors. Thus, for the general case, we hypothesize that the upper limit of the $\prod$-product is $\lceil \frac{n}{m} \rceil$.

It is important to note that in Eq. 5.3.3, the limits range from $k = 0$ to $\lceil \frac{n}{m} \rceil - 1$, whereas the rising product definition we are seeking should have limits from $k = 1$ to $\lceil \frac{n}{m} \rceil$. This shift in counting from 1 instead of 0 affects only the way factors are counted, without altering the final result.

Let's try to generalize from Eq. 2.4.5. Before extending this to higher orders, it is wise to first consider the traditional factorial by substituting 1 for every instance of 2. This will help us verify if our approach is correct before proceeding further.

Now, Eq. 2.4.1 becomes (for $n \in \mathbb{Z}_0^+$):

$$n!_{(1)} = \prod_{k=1}^{\lceil \frac{n}{1} \rceil} (1k - n \bmod 1) = \prod_{k=1}^{\lceil n \rceil} (k - 0) = \prod_{k=1}^{n} k \,. \quad \text{(Eq. 5.4.1)}$$

This reduction indeed returns us to the original formulation of the rising product for the traditional factorial. With this confirmation, let's now express the rising product for $m = 3$:

$$n!_{(3)} = \prod_{k=1}^{\lceil \frac{n}{3} \rceil} (3k - n \bmod 3) \,, \quad n \in \mathbb{Z}^+ . \quad \text{(Eq. 5.4.2)}$$

The following examples illustrate its behavior:

$$6!_{(3)} = \prod_{k=1}^{\lceil \frac{6}{3} \rceil} (3k - 6 \bmod 3) = \prod_{k=1}^{2} (3k - 0) = 3 \cdot 6 \,,$$

$$7!_{(3)} = \prod_{k=1}^{\lceil \frac{7}{3} \rceil} (3k - 7 \bmod 3) = \prod_{k=1}^{3} (3k - 1) = 2 \cdot 5 \cdot 8 \,.$$

Unfortunately, there appears to be an error in our formulation. The product for $7!_{(3)}$ should be $1 \cdot 4 \cdot 7$, not $2 \cdot 5 \cdot 8$. The value of $6!_{(3)}$ was calculated correctly, though this seems to be coincidental.

To identify the mistake, let's examine a few more examples:

$$8!_{(3)} = \prod_{k=1}^{\lceil \frac{8}{3} \rceil} (3k - 8 \bmod 3) = \prod_{k=1}^{3} (3k - 2) = 1 \cdot 4 \cdot 7 \,,$$

$$9!_{(3)} = \prod_{k=1}^{\lceil \frac{9}{3} \rceil} (3k - 9 \bmod 3) = \prod_{k=1}^{3} (3k - 0) = 3 \cdot 6 \cdot 9 \,.$$

Again, there's a pattern where products for multiples of 3, such as $6!_{(3)}$ and $9!_{(3)}$, are computed correctly, while the others are not. Notably, the product for $7!_{(3)}$ corresponds to that of $8!_{(3)}$ and vice versa.

Given the complexity of this issue, let's now examine the case for $m = 4$ to see if any new patterns emerge.

The incorrect definition of Eq. 5.4.2 for $m = 4$ is:

$$n!_{(4)} = \prod_{k=1}^{\lceil \frac{n}{4} \rceil} (4k - n \bmod 4) \,, \quad n \in \mathbb{Z}^+ . \quad \text{(Eq. 5.4.3)}$$

Let's go through some examples:

$$5!_{(4)} = \prod_{k=1}^{\lceil \frac{5}{4} \rceil} (4k - 5 \bmod 4) = \prod_{k=1}^{2} (4k - 1) = 3 \cdot 7 \,,$$

$$6!_{(4)} = \prod_{k=1}^{\lceil \frac{6}{4} \rceil} (4k - 6 \bmod 4) = \prod_{k=1}^{2} (4k - 2) = 2 \cdot 6 \,,$$

$$7!_{(4)} = \prod_{k=1}^{\lceil \frac{7}{4} \rceil} (4k - 7 \bmod 4) = \prod_{k=1}^{2} (4k - 3) = 1 \cdot 5 \,,$$

$$8!_{(4)} = \prod_{k=1}^{\lceil \frac{8}{4} \rceil} (4k - 8 \bmod 4) = \prod_{k=1}^{2} (4k - 0) = 4 \cdot 8 \,.$$

The products for $6!_{(4)}$ and $8!_{(4)}$ appear to be correct, while those for $7!_{(4)}$ and $9!_{(4)}$ are not. Specifically, these multifactorials seem to produce each other's result.

To understand why, consider this: for $5!_{(4)}$, we would want its rising product to be $1 \cdot 5$, or $(4k - 3)$ in the examples above. For $7!_{(4)}$, its rising product is $3 \cdot 7$ and so it could be expressed by the terms $(4k - 1)$.

This suggests that the expression $n \bmod 4$ needs to be modified. In order to do that, let's name the expression a function of n and m, for instance $f(n, 4)$. Let's also create a table comparing its desired output to our current incorrect expression:

n	1	2	3	4	5	6	7	8	9	10
$n \bmod 4$	1	2	3	0	1	2	3	0	1	2
$f(n, 4)$	3	2	1	0	3	2	1	0	3	2

Tbl. 5.4.1: The function $f(n, 4)$ in the case of the multifactorial $m = 4$ as a rising product

There is a pattern here, connecting $n \bmod 4$ with $f(n, 4)$. They match on even integers, which is confirmed by our examples of $6!_{(4)}$ and $8!_{(4)}$. They were found to be calculated correctly, where as odd integers were not.

As n increases, the expression $n \bmod 4$ cycles through 0, 1, 2, and 3. The function $f(n, 4)$ however, seems to exhibit an opposite pattern, falling from 3 to 0 and then rising back up to 3.

Let's try to replicate that behavior. One way to mirror the outputs of $n \bmod 4$ is to subtract it from the number 4:

n	1	2	3	4	5	6	7	8	9
$n \bmod 4$	1	2	3	0	1	2	3	0	1
$4 - n \bmod 4$	3	2	1	4	3	2	1	4	3
$f(n,4)$	3	2	1	0	3	2	1	0	3

Tbl. 5.4.2: The expression $(4 - n \bmod 4)$ in the case of the multifactorial $m = 4$ as a rising product

We're getting closer to finding $f(n,4)$, but there's a small problem. When n is a multiple of 4, $f(n,4)$ should be 0, but $4 - n \bmod 4$ gives us 4. We need to change this so that 4 becomes 0, while keeping 1, 2, and 3 the same.

Luckily, we do not need any new methodologies to solve this issue. We can apply mod 4 to our current expression for $f(n,4)$, resulting in the nested expression $(4 - n \bmod 4) \bmod 4$. This gives us 0 when n is a multiple of 4, but doesn't alter the other numbers.

In other words, using mod 4 again doesn't change anything if the result is 1, 2, or 3. But when n is 4, here's what happens:

$$(4 - 4 \bmod 4) \bmod 4 = (4 - 0) \bmod 4 = 0\,.$$

This does exactly what we want. We can make it simpler using a rule of modular arithmetic:

$$\begin{aligned}(4 - n \bmod 4) \bmod 4 &\rightarrow (4 - n) \bmod 4\\ &\rightarrow (-n) \bmod 4\,.\end{aligned}$$

No matter what $n \bmod 4$ gives us, using mod 4 again won't change the result. This is because mod 4 is idempotent[14], so we can remove the inner mod 4 and just write $(4-n) \bmod 4$. We can also remove the 4 that we're subtracting n from, because it doesn't change the final answer.

Now that we have found $f(n,4)$, let's include it in Tbl. 5.4.2 for completeness:

n	1	2	3	4	5	6	7	8
$n \bmod 4$	1	2	3	0	1	2	3	0
$4 - (n \bmod 4)$	3	2	1	4	3	2	1	4
$(-n) \bmod 4$	3	2	1	0	3	2	1	0
$f(n,4)$	3	2	1	0	3	2	1	0

Tbl. 5.4.3: The expression $(-n) \bmod 4$ in the case of the multifactorial $m = 4$ as a rising product

In short, we have solved the issue of the miscalculated multifactorials by investigating the case of $m = 4$. The resulting rising product definition of the quadruple factorial is as follows:

$$n!_{(4)} = \prod_{k=1}^{\lceil \frac{n}{4} \rceil} \big(4k - (-n) \bmod 4\big)\,, \quad n \in \mathbb{Z}^+. \tag{Eq. 5.4.4}$$

Before we make this work for all cases, let's check if it fixes the problem for $m = 3$ and if it still works for $m = 2$ and $m = 1$.

For $m = 3$, the rising product definition looks like this:

$$n!_{(3)} = \prod_{k=1}^{\lceil \frac{n}{3} \rceil} \big(3k - (-n) \bmod 3\big)\,, \quad n \in \mathbb{Z}^+. \tag{Eq. 5.4.5}$$

It is exemplified below:

$$6!_{(3)} = \prod_{k=1}^{\lceil \frac{6}{3} \rceil} \big(3k - (-6) \bmod 3\big) = \prod_{k=1}^{2} (3k - 0) = \cdots$$

$$7!_{(3)} = \prod_{k=1}^{\lceil \frac{7}{3} \rceil} \big(3k - (-7) \bmod 3\big) = \prod_{k=1}^{3} (3k - 2) = \cdots$$

$$8!_{(3)} = \prod_{k=1}^{\lceil \frac{8}{3} \rceil} \big(3k - (-8) \bmod 3\big) = \prod_{k=1}^{3} (3k - 1) = \cdots$$

$$9!_{(3)} = \prod_{k=1}^{\lceil \frac{9}{3} \rceil} \big(3k - (-9) \bmod 3\big) = \prod_{k=1}^{3} (3k - 0) = \cdots$$

You can check for yourself that these products give the correct rising product of $n!_{(3)}$. We don't need to show all the calculations in this paper, but it's a good exercise to work through them on your own.

Let's now check what happens for $m = 2$:

$$\begin{aligned}(2 - n \bmod 2) \bmod 2 &\rightarrow (2 - n) \bmod 2\\ &\rightarrow (-n) \bmod 2 \rightarrow n \bmod 2\,.\end{aligned}$$

This works out nicely because $n \bmod 2$ always gives either 0 or 1, no matter if n is positive or negative. It only cares about whether n can be divided by 2 without a remainder. Interestingly, this simple behavior doesn't hold true for larger values of m, as we saw earlier.

Finally, let's check what happens when $m = 1$:

$$\begin{aligned}(1 - n \bmod 1) \bmod 1 &\rightarrow (1 - 0) \bmod 1\\ &\rightarrow 1 \bmod 1 \rightarrow 0\,.\end{aligned}$$

Indeed, this result is correct since the modular term is absent from the rising product of $n!_1$.

Thus, now that we've checked all these cases, we're ready to write down the full definition of the multi-

[14] Idempotence is the property of certain operations in mathematics and computer science whereby they can be applied multiple times without changing the result beyond the initial application [5].

factorial as a rising product. Here it is:

$$n!_{(m)} = \prod_{k=1}^{\left\lceil \frac{n}{m} \right\rceil} \big(mk - (-n) \bmod m\big)\,, \quad n \in \mathbb{Z}^+. \tag{Eq. 5.4.6}$$

The upper limit of this $\prod$-product looks a lot like the one we used for the falling product definition of the multifactorial. Because of this similarity, we can make our new equation work for even more numbers by utilizing the empty product. This lets us define the equation for $\{n \in \mathbb{Z} \,|\, -m < n\}$.

The trick about nested modular arithmetic was not obvious earlier, because we didn't look closely at factorials of higher orders on their own. When we were working with the falling product, our first attempt at generalizing the double factorial definition worked out perfectly. But when we tried to do the same thing for the rising product definition of the multifactorial, we found out that our initial approach was flawed from the start.

This process of trying things out and sometimes getting it wrong is a crucial part of how mathematics works. It's important to show the whole journey of discovery, not just list the final results. The main goal of this paper isn't to just present a new way of defining factorials, but to provide a peek into the thought process that got us there.

The final result on its own isn't worth much compared to the careful, step-by-step process we used to find it. That's why this series of papers spends time looking at the parts of math that don't work out, not just the successes.

5.5 Synopsis

To sum up, Section 5 was about introducing the concept of multifactorials and developing various definitions that describe it. We expanded its domain to negative integers down to $n = -m + 1$ to simplify its recursive formulation, and found that the $\prod$-products can be also extended to this domain as well.

Let's wrap up this section by listing the different ways we've defined the multifactorial.

First, we have the recursive definition:

$$n!_{(m)} = n(n-m)!_{(m)}\,, \quad n \in \mathbb{Z}^+, \tag{Eq. 5.2.6}$$

where

$$n!_{(m)} = 1\,, \quad \{n \in \mathbb{Z} \,|\, -m < n \leq 0\}\,. \tag{Eq. 5.2.7}$$

Next, we have the falling product definition:

$$n!_{(m)} = \prod_{k=0}^{\left\lceil \frac{n}{m} \right\rceil - 1} (n - mk)\,, \quad \{n \in \mathbb{Z} \,|\, -m < n\}\,. \tag{Eq. 5.3.4}$$

Finally, the rising product definition of the multifactorial is defined as:

$$n!_{(m)} = \prod_{k=1}^{\left\lceil \frac{n}{m} \right\rceil} \big(mk - (-n) \bmod m\big)\,, \tag{Eq. 5.4.6}$$

where

$$\{n \in \mathbb{Z} \,|\, -m < n\}\,. \tag{Eq. 5.5.1}$$

These definitions summarize what we've learned about multifactorials so far. Now we're ready to look at how multifactorials extend to all negative integers.

6 Omnifactorial

6.1 Introduction

This section extends multifactorials to include all integers, introducing the *omnifactorial* ($n!_m$), a term coined in this paper. We will first determine omnifactorials for all negative integers, then define them in four different ways: recursively, as a Roman-like expression, and with two $\prod$-product definitions. We will focus on the triple factorial before generalizing the concept to all omnifactorials.

We will start by focusing on the triple factorial and then briefly explore factorials of order 4 before generalizing the concept to all omnifactorials.

6.2 Negative non-multiples of m

To begin, let's expand the domain where multifactorials are defined by using their recursive relation, similar to the approach used for negative odd $n!_2$ in Subsection 3.2.

The recursive property that links one multifactorial to a larger one is iterated as follows:

$$n!_{(m)} = \frac{(n+m)!_{(m)}}{n+m}\,, \quad n \in \mathbb{Z}_0^+. \qquad \text{(Eq. 5.1.3)}$$

For factorials of order 3, the relation becomes:

$$n!_{(3)} = \frac{(n+3)!_{(3)}}{n+3}\,, \quad n \in \mathbb{Z}_0^+. \qquad \text{(Eq. 5.2.3)}$$

This equation is defined for non-negative integers, but our goal is to find factorials of order 3 for negative integers. Let's evaluate Eq. 5.2.3 for a few negative numbers to see what happens:

$$(-1)!_3 = \frac{(-1+3)!_3}{-1+3} = \frac{2!_3}{2} = 1\,,$$

$$(-2)!_3 = \frac{(-2+3)!_3}{-2+3} = \frac{1!_3}{1} = 1\,,$$

$$(-3)!_3 = \frac{(-3+3)!_3}{-3+3} = \frac{0!_3}{0} = \frac{1}{0}\,.$$

It seems that $(-1)!_3$ and $(-2)!_3$ both equal to 1, but $(-3)!_3$ is undefined.

It's important to note that extending the domain of a recursive relation in this manner is just an application of Eq. 5.2.3 to a broader domain, assuming that negative factorials of order 3 exist and are assigned certain values.

For now, let's set aside $(-3)!_3$ and proceed with the recursive property:

$$(-4)!_3 = \frac{(-4+3)!_3}{-4+3} = \frac{(-1)!_3}{-1} = -1\,,$$

$$(-5)!_3 = \frac{(-5+3)!_3}{-5+3} = \frac{(-2)!_3}{-2} = -\frac{1}{2}\,,$$

$$(-6)!_3 = \frac{(-6+3)!_3}{-6+3} = \frac{(-3)!_3}{-3} = -\frac{1/0}{3}\,.$$

Clearly, $(-6)!_3$ is undefined because it directly relates to $(-3)!_3$. However, negative integers that are not multiples of 3 seem to have no issues with their calculation.

This behavior is illustrated by the figure below:

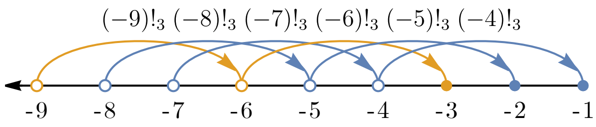

Fig. 6.2.1: Undefined values of $n!_3$

The blue lines indicate relationships with larger factorials of order 3, while the orange arrows show where the values are undefined due to division by 0.

Let's gather the factorials of order 3 we've identified in a table to look for possible patterns:

n	-9	-8	-7	-6	-5	-4	-3	-2	-1
$n!_3$	–	1/10	1/4	–	-1/2	-1	–	1	1

Tbl. 6.2.1: $n!_3$ with undefined values

We can adjust this table to emphasize patterns that aren't immediately obvious. First, we'll remove negative multiples of 3 since they currently lack a definition. Then, we'll split the table into two, each containing a different subset of non-multiples of 3. The first table includes negative factorials of order 3 in which $n = 3k + 1$:

n	-14	-11	-8	-5	-2	1	4	7
$n!_3$	1/880	-1/80	1/10	-1/2	1	1	4	28

Tbl. 6.2.2: $n!_3$ where $n = 3k + 1$

The second table includes $n!_3$ where $n = 3k + 2$:

n	-13	-10	-7	-4	-1	2	5	8
$n!_3$	1/280	-1/28	1/4	-1	1	2	10	80

Tbl. 6.2.3: $n!_3$ where $n = 3k + 2$

We observe that negative factorials of order 3 alternate signs as they decrease by 3, and that they are reciprocals of positive triple factorials with an offset of 3.

To better understand these patterns, let's edit the previous tables by highlighting the positive and negative signs and noting the triple factorials in the denominators. This approach is taken from our analysis of negative odd factorials of order 2, back in Subsection 3.2.

The resulting table for $n = 3k+1$ is shown below:

n	-14	-11	-8	-5
$n!_3$	$(+1)/11!_3$	$(-1)/8!_3$	$(+1)/5!_3$	$(-1)/2!_3$

Tbl. 6.2.4: $n!_3$ where $n = 3k + 1$ with highlighted components

The other subset, $n = 3k + 2$, becomes:

n	-13	-10	-7	-4
$n!_3$	$(+1)/10!_3$	$(-1)/7!_3$	$(+1)/4!_3$	$(-1)/1!_3$

Tbl. 6.2.5: $n!_3$ where $n = 3k + 2$ with highlighted components

There is a clear pattern here, one that can easily be formulated into a precise mathematical expression.

Given that the values presented in the tables above are fractions, we can state that the negative factorials of order 3 are given by the following formula:

$$n!_3 = \frac{f(n,3)}{g(n,3)}\,, \quad \{n, k \in \mathbb{Z} \,|\, n \neq 3k\}\,, \quad \text{(Eq. 6.2.1)}$$

where $f(n, 3)$ and $g(n, 3)$ are functions of n. They will later be generalized to $f(n, m)$ and $g(n, m)$ respectively.

Let's first focus on the denominator function, $g(n, 3)$. As done previously, we will create a table to display the outputs that both functions should yield, corresponding to different values of n:

n	-8	-7	-5	-4
$n!_3$	$(+1)/5!_3$	$(+1)/4!_3$	$(-1)/2!_3$	$(-1)/1!_3$
$f(n, 3)$	$+1$	$+1$	-1	-1
$g(n, 3)$	$5!_3$	$4!_3$	$2!_3$	$1!_3$

Tbl. 6.2.6: Negative factorials of order 3 in terms of $f(n, 3)$ and $g(n, 3)$

From the table, it is clear that $g(n, 3)$ is:

$$g(n, 3) = (-n-3)!_3\,, \quad \{n, k \in \mathbb{Z} \,|\, n \neq 3k\}\,. \quad \text{(Eq. 6.2.2)}$$

To fully describe the negative factorials of order 3 that are not multiples of 3, we now only need to determine the formula for $f(n, 3)$.

We will proceed by following the same methodology employed earlier in this paper when analyzing the double factorial. Rather than directly deriving $f(n, 3)$, we will first compute $f(k, 3)$ for values of $n = 3k + 1$ and $n = 3k + 2$.

To start with, let's focus on the case where $n = 3k + 1$. The following table includes the variables n, k, and their corresponding values of $f(k, 3)$:

n	-14	-11	-8	-5
k	-5	-4	-3	-2
$f(k, 3)$	$+1$	-1	$+1$	-1

Tbl. 6.2.7: The function $f(k, 3)$ in negative $n!_3$ where $n = 3k + 1$

Fortunately, there is no need to derive the expression for $f(k, 3)$ from scratch. Tbl. 3.2.4 contains identical values of k and the corresponding outputs of what was then referred to as $f(k)$. The function identified at that time was:

$$f(k) = (-1)^{k+1}\,, \quad k \in \mathbb{Z}^- \setminus \{-1\}\,. \quad \text{(Eq. 3.2.7)}$$

Thus, the function we are looking for in this case will have exactly the same form:

$$f(k, 3) = (-1)^{k+1}\,, \quad \{k \in \mathbb{Z} \,|\, k < \text{-}1\}\,. \quad \text{(Eq. 6.2.3)}$$

Now, we can transition back to the variable n. In our current analysis, n is equivalent to $3k + 1$, meaning that $k = \frac{n-1}{3}$. Substituting this expression into Eq. 6.2.3, we arrive at:

$$f(n, 3) = (-1)^{\frac{n+2}{3}}\,, \quad \{k \in \mathbb{Z} \,|\, k < \text{-}1\}\,. \quad \text{(Eq. 6.2.4)}$$

This leads to a definition for negative factorials of order 3 that are 1 above multiples of 3:

$$n!_3 = \frac{(-1)^{\frac{n+2}{3}}}{(-n-3)!_{(3)}}\,, \quad \{n = 3k + 1 \,|\, k \in \mathbb{Z} \,|\, k < \text{-}1\}\,. \quad \text{(Eq. 6.2.5)}$$

Having established this formula, we can now seek an equivalent expression for $n = 3k+2$. Fortunately, the analysis required here will be relatively concise.

To begin, we will construct another table, this time including values of k and the corresponding values of $f(k, 3)$ for these new cases of n:

n	-13	-10	-7	-4
k	-5	-4	-3	-2
$f(k, 3)$	$+1$	-1	$+1$	-1

Tbl. 6.2.8: The function $f(k, 3)$ in negative $n!_3$ where $n = 3k + 2$

Coincidentally, the table shown above is identical to the one presented in Tbl. 6.2.7. This indicates that the function $f(k, 3)$ is the same as before, for the case where $n = 3k + 2$:

$$f(k, 3) = (-1)^{k+1}\,, \quad \{k \in \mathbb{Z} \,|\, k < \text{-}1\}\,. \quad \text{(Eq. 6.2.6)}$$

By following a similar set of substitution steps as we did previously, we can derive the definition that describes $n!_3$ for the scenario where $n = 3k + 2$.

Specifically, reverting from k back to n using the relationship $k = \frac{n-2}{3}$, yields:

$$n!_3 = \frac{(-1)^{\frac{n+1}{3}}}{(-n-3)!_{(3)}}\,, \quad \{n = 3k+2 \,|\, k \in \mathbb{Z} \,|\, k < \text{-1}\}\,. \tag{Eq. 6.2.7}$$

With this, we conclude our analysis of factorials of order 3. However, our broader objective is to explore the general case. Although we could perform a similar detailed analysis for factorials of higher orders, it is not necessary to repeat the process exhaustively in this paper.

Instead, we list the resulting definitions that would emerge if we considered factorials of order 4. These are as follows:

$$n!_4 = \frac{(-1)^{\frac{n+3}{4}}}{(-n-4)!_{(4)}}\,, \quad \{n = 4k+1 \,|\, k \in \mathbb{Z} \,|\, k < \text{-1}\}\,. \tag{Eq. 6.2.8}$$

$$n!_4 = \frac{(-1)^{\frac{n+2}{4}}}{(-n-4)!_{(4)}}\,, \quad \{n = 4k+2 \,|\, k \in \mathbb{Z} \,|\, k < \text{-1}\}\,. \tag{Eq. 6.2.9}$$

$$n!_4 = \frac{(-1)^{\frac{n+1}{4}}}{(-n-4)!_{(4)}}\,, \quad \{n = 4k+3 \,|\, k \in \mathbb{Z} \,|\, k < \text{-1}\}\,. \tag{Eq. 6.2.10}$$

Now, let's attempt to identify a connection between these expressions. It seems that for $n = 4k+l$, where l takes on the values 1, 2, or 3, the only aspect that changes in the definitions above is the exponent of (-1) in the numerator.

We can define this exponent as a function of n, which we'll call $h(n, 4)$. The table below indicates what this new function would be for the different values of l:

l	1	2	3
$h(n,4)$	$\frac{n+3}{4}$	$\frac{n+2}{4}$	$\frac{n+1}{4}$

Tbl. 6.2.9: The function $h(n, 4)$ in negative $n!_4$ where $n = 4k + l$

It appears that we need to add something to $\frac{n}{4}$ that decreases as l increases. Notice that the sum $\frac{l}{4} + h(n, 4)$ remains constant:

l	1	2	3
$h(n,4)$	$\frac{n+3}{4}$	$\frac{n+2}{4}$	$\frac{n+1}{4}$
$\frac{l}{4} + h(n,4)$	$\frac{n+4}{4}$	$\frac{n+4}{4}$	$\frac{n+4}{4}$

Tbl. 6.2.10: The sum $\frac{l}{4} + h(n, 4)$ in negative $n!_4$ where $n = 4k + l$

In other words, we have discovered that:

$$\frac{l}{4} + h(n,4) = \frac{n}{4} + 1\,.$$

By solving for $h(n, 4)$, we arrive at an expression for this function:

$$h(n,4) = \frac{n-l}{4} + 1\,, \quad l = \{1,2,3\}\,.$$

However, the variable l is not a proper independent variable in this context and needs to be converted into a function of n.

Recall that l was initially defined as the number added to $4k$ to express n. Given that $n = 4k + l$, we can apply modular arithmetic to this equation and solve[15] for l:

$$n = 4k + l \Rightarrow n \bmod 4 = (4k+l) \bmod 4$$
$$\Rightarrow n \bmod 4 = l\,.$$

Therefore, $h(n, 4)$ becomes:

$$h(n,4) = \frac{n - n \bmod 4}{4} + 1\,. \tag{Eq. 6.2.11}$$

We have now established a generalized formula that describes negative factorials of order 4 that are not multiples of 4. This expression essentially combines the definitions Eq. 6.2.8 through Eq. 6.2.10 into a single expression:

$$n!_4 = \frac{(-1)^{\frac{n-n \bmod 4}{4}+1}}{(-n-4)!_{(4)}}\,, \quad \{n, k \in \mathbb{Z}^- \,|\, n \neq 4k\}\,. \tag{Eq. 6.2.12}$$

Before generalizing this to the broader case of the omnifactorial, we need to verify whether such an expression would also work for $m = 2$. If it does, then Eq. 6.2.11 must align with the corresponding expression in Eq. 3.2.9.

In other words, if we replaced instances of the number 4 in Eq. 6.2.11 with 2, it should match the expression $\frac{n+1}{2}$ for odd numbers. Let's verify:

$$\frac{n - n \bmod 2}{2} + 1 = \frac{n-1}{2} + 1 = \frac{n+1}{2}\,, \quad n \text{ is } \textit{odd}\,.$$

Indeed, the expression holds true. A quick check for factorials of order 3 also yields the definitions we derived earlier in Eq. 6.2.5 and Eq. 6.2.7.

In conclusion, we can now state the general formula for the negative omnifactorials that are not multiples of m:

$$n!_m = \frac{(-1)^{\frac{n-n \bmod m}{m}+1}}{(-n-m)!_{(m)}}\,, \quad \{n, k \in \mathbb{Z}^- \,|\, n \neq mk\}\,. \tag{Eq. 6.2.13}$$

In summary, we have established a formula that applies to $n!_m$ when $n \neq mk$. In the next subsection, we will determine the values for those undefined omnifactorials that we initially skipped. These expressions will be unified in Subsection 6.4, culminating in the Roman-like definition of the omnifactorial.

[15] Actually, we can only solve for $l \bmod 4$, but since l is set as an integer between 1 and 3, then $l \bmod 4 = l$.

6.3 Negative multiples of m

Now that we have a clear definition for negative omnifactorials, it's time to explore the undefined values we previously skipped.

We'll begin by examining what negative omnifactorial multiples of m might look like, using the known cases for factorials of order 1 and 2 as our foundation. Following this, we'll work towards defining them in a manner similar to how we defined non-multiples of m.

Let's start with factorials of order 3, as they represent the next simplest unexplored case. Here are the values we discovered by extending the domain of $n!_3$ using their recursive property:

n	-11	-10	-8	-7	-5	-4	-2	-1
$n!_3$	-1/80	-1/28	1/10	1/4	-1/2	-1	1	1

Tbl. 6.3.1: $n!_3$ without undefined values

As we've seen before, they are reciprocals of triple factorials with alternating signs. To better understand this pattern, let's expand their product:

$$(-10)!_3 = (-10) \cdot \frac{1}{-10} \cdot \frac{1}{-7} \cdot \frac{1}{-4} \cdot \frac{1}{-1} = -\frac{1}{28} \,,$$

$$(-11)!_3 = (-11) \cdot \frac{1}{-11} \cdot \frac{1}{-8} \cdot \frac{1}{-5} \cdot \frac{1}{-2} = -\frac{1}{80} \,.$$

Taking this a step further, we would get:

$$(-12)!_3 = (-12) \cdot \frac{1}{-12} \cdot \frac{1}{-9} \cdot \frac{1}{-6} \cdot \frac{1}{-3} = -\frac{1}{162} \,.$$

It's important to note that $(-12)!_3$ is currently undefined in our existing framework. Up to this point, we've explored negative omnifactorials by expanding their recursive properties, which fail to evaluate at certain numbers. This approach led us to identify negative omnifactorials that are not multiples of m.

We propose that these examples demonstrate a method for determining values for the previously undefined factorials. While we don't yet have established values for them, we suggest using this approach to calculate them. This reasoning aligns with the logic we applied to negative even double factorials.

Let's see what other values of $n!_3$ can be found in this manner:

$$(-12)!_3 = (-12) \cdot \frac{1}{-12} \cdot \frac{1}{-9} \cdot \frac{1}{-6} \cdot \frac{1}{-3} = -\frac{1}{162} \,,$$

$$(-9)!_3 = (-9) \cdot \frac{1}{-9} \cdot \frac{1}{-6} \cdot \frac{1}{-3} = \frac{1}{18} \,,$$

$$(-6)!_3 = (-6) \cdot \frac{1}{-6} \cdot \frac{1}{-3} = -\frac{1}{3} \,,$$

$$(-3)!_3 = (-3) \cdot \frac{1}{-3} = 1 \,.$$

Interestingly, these are also reciprocals of positive triple factorials, with alternating signs. This pattern applies to all omnifactorials, regardless of the order m.

The following table gathers them all in one place:

n	-15	-12	-9	-6	-3
$n!_3$	1/1944	-1/162	1/18	-1/3	1

Tbl. 6.3.2: Negative $n!_3$ where $n = 3k$

Now that we have these values, our next step is to formulate a definition that generates them. We'll approach this systematically like before, beginning by expressing these factorials as:

$$n!_3 = \frac{f(n,3)}{g(n,3)} \,, \quad \{n, k \in \mathbb{Z} \,|\, n = 3k\} \,, \quad \text{(Eq. 6.3.1)}$$

where $f(n,3)$ and $g(n,3)$ are functions we'll determine shortly.

It's clear that $g(n,3)$ is the same as before:

$$g(n,3) = (-n-3)!_3 \,, \quad \{n, k \in \mathbb{Z} \,|\, n = 3k\} \,. \quad \text{(Eq. 6.3.2)}$$

Before proceeding, let's take a moment to look at omnifactorials in more detail. The large table below shows $n!_m$ values up to $m = 5$, giving us a wider perspective on these variations of the factorial. It is by seeing the bigger picture that we can uncover insights, helpful in our goals.

n	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
$n!_1$	-1/5040	1/720	-1/120	1/24	-1/6	1/2	-1	1	1	1	2	6	24	120	720	5040	40320
$n!_2$	-1/48	-1/15	1/8	1/3	-1/2	-1	1	1	1	1	2	3	8	15	48	105	384
$n!_3$	1/10	1/4	-1/3	-1/2	-1	1	1	1	1	1	2	3	4	10	18	28	80
$n!_4$	-1/4	-1/3	-1/2	-1	1	1	1	1	1	1	2	3	4	5	12	21	32
$n!_5$	-1/3	-1/2	-1	1	1	1	1	1	1	1	2	3	4	5	6	14	24

Tbl. 6.3.3: Omnifactorials

Let's now return to our main task. We'll now set $n = 3k$ where k is an integer, and create this table:

n	-12	-9	-6	-3
$n!_3$	$(-1)/9!_3$	$(+1)/6!_3$	$(-1)/3!_3$	$(+1)/0!_3$
k	-4	-3	-2	-1
$f(k,3)$	-1	$+1$	-1	$+1$

Tbl. 6.3.4: Negative $n!_3$ in terms of $f(k,3)$

At this point, the pattern should be clear enough to predict what comes next. We can see that $f(k,3) = (-1)^{k+1}$, and by setting $k = \frac{n}{3}$ we arrive at our desired definition:

$$n!_3 = \frac{(-1)^{\frac{n}{3}+1}}{(-n-3)!_{(3)}}, \quad \{n = 3k \,|\, k \in \mathbb{Z}^-\}. \qquad \text{(Eq. 6.3.3)}$$

This process can be repeated for factorials of higher orders. For $m = 4$, we would get:

$$n!_4 = \frac{(-1)^{\frac{n}{4}+1}}{(-n-4)!_{(4)}}, \quad \{n = 4k \,|\, k \in \mathbb{Z}^-\}. \qquad \text{(Eq. 6.3.4)}$$

You are encouraged to verify this on your own. The general case turns out to be of the form:

$$n!_m = \frac{(-1)^{\frac{n}{m}+1}}{(-n-m)!_{(m)}}, \quad \{n = mk \,|\, k \in \mathbb{Z}^-\}. \qquad \text{(Eq. 6.3.5)}$$

Since this result comes from intuition, we should check it for small m values that match known definitions. For $m = 1$, we get:

$$n!_1 = \frac{(-1)^{\frac{n}{1}+1}}{(-n-1)!_{(1)}}, \quad n \in \mathbb{Z}^-, \qquad \text{(Eq. 6.3.6)}$$

which is similar to the original definition of the Roman factorial:

$$\lfloor n \rceil! = \frac{(-1)^{-n-1}}{(-n-1)!}, \quad n \in \mathbb{Z}^-. \qquad \text{(Eq. 1.5.1)}$$

In fact, they only differ in the exponent of the numerator. However, upon closer inspection, we find they are actually equivalent:

$$(-1)^{-n-1} = \frac{1}{(-1)^{-(-n-1)}} = \frac{1}{(-1)^{n+1}} = (-1)^{n+1}.$$

Now, let's examine the case of $m = 2$:

$$n!_2 = \frac{(-1)^{\frac{n}{2}+1}}{(-n-2)!_{(2)}}, \quad \{n = 2k \,|\, k \in \mathbb{Z}^-\}. \qquad \text{(Eq. 6.3.7)}$$

Indeed, this matches exactly what we found earlier in Subsection 3.3:

$$n!_2 = \frac{(-1)^{\frac{n}{2}+1}}{(-n-2)!_{(2)}}, \quad n \in \mathbb{Z}^-_{even}. \qquad \text{(Eq. 3.3.5)}$$

This verification concludes our exploration of negative multiples of m. It's worth noting that this subsection was concise, which suggests our intuition was on target. Through our in-depth analysis of the procedures in this study, we've developed skills to better assess how to formulate generalized definitions.

6.4 Roman-like definition

Let's now attempt to unify the expressions that describe omnifactorials for negative integers. For non-multiples of m, the omnifactorial is given by:

$$n!_m = \frac{(-1)^{\frac{n - n \bmod m}{m}+1}}{(-n-m)!_{(m)}}, \quad \{n, k \in \mathbb{Z}^- \,|\, n \neq mk\}. \qquad \text{(Eq. 6.2.13)}$$

For cases where $n = mk$, the definition is slightly different:

$$n!_m = \frac{(-1)^{\frac{n}{m}+1}}{(-n-m)!_{(m)}}, \quad \{n = mk \,|\, k \in \mathbb{Z}^-\}. \qquad \text{(Eq. 6.3.5)}$$

At first glance, the two expressions differ only in the exponent in the numerator. To better understand this difference, let's compare them directly:

$n \neq mk$	$n = mk$
$\frac{n - n \bmod m}{m} + 1$	$\frac{n}{m} + 1$

Tbl. 6.4.1: The exponents of the numerators in the Roman-like definitions of the omnifactorial

To further simplify, we can eliminate the common terms and focus on what differs. By subtracting the term $\frac{n}{m} + 1$ from both cases, we get:

$n \neq mk$	$n = mk$
$-\frac{n \bmod m}{m}$	0

Tbl. 6.4.2: The discrepancies of the numerators in the Roman-like definitions of the omnifactorial

There are multiple ways to reconcile these two terms, such as introducing a *F.F.* that becomes 0 whenever $n = mk$. However, there's a more straightforward approach.

Consider what happens to the term $\frac{n \bmod m}{m}$ when $n = mk$:

$$\frac{n \bmod m}{m} \to \frac{mk \bmod m}{m} = \frac{0}{m} = 0.$$

Interestingly, this means that the expression initially defined for non-multiples of m also works for multiples of m without requiring any modifications.

This result wasn't premeditated, as the methodology of this work involves deriving results through a structured process rather than guessing or forcing outcomes. In this case, the process led us to a unified result that required no further adjustments. Had the analysis been more complex, it would have been fully documented, as we did when our assumptions didn't hold in Subsection 5.4.

In conclusion, we can generalize the definition of negative omnifactorials as follows:

$$n!_m = \frac{(-1)^{\frac{n - n \bmod m}{m}+1}}{(-n-m)!_{(m)}}\,, \quad n \in \mathbb{Z}^- . \qquad \text{(Eq. 6.4.1)}$$

Thus, the piece-wise Roman-like definition of the omnifactorial, including the case for $n \in \mathbb{Z}_0^+$, is:

$$\lfloor n \rceil !_m = \begin{cases} n!_{(m)} & ,\ n \in \mathbb{Z}_0^+ \\ \dfrac{(-1)^{\frac{n - n \bmod m}{m}+1}}{(-n-m)!_{(m)}} & ,\ n \in \mathbb{Z}^- , \end{cases} \qquad \text{(Eq. 6.4.2)}$$

where the multifactorial is recursively defined as:

$$n!_{(m)} = n(n-m)!_{(m)}\,, \quad n \in \mathbb{Z}^+, \qquad \text{(Eq. 5.2.6)}$$

with the base case

$$n!_{(m)} = 1\,, \quad \{n \in \mathbb{Z} \,|\, -m < n \leq 0\}\,. \qquad \text{(Eq. 5.2.7)}$$

Eq. 6.4.2 successfully completes our goal for this subsection. The two cases of the definition will be unified later in Section 7 in the same way as the factorials of orders 1 and 2 were generalized.

6.5 Recursive definition

In this subsection, we aim to establish the recursive definition of the multifactorial. While it has already been defined for positive integers, Eq. 5.2.6 extends it to include values down to $(-m+1)!_m$. However, we now need to define omnifactorial values that fall outside this range.

Let's begin by revisiting the recursive definition of the Roman factorial:

$$n!_1 = \begin{cases} n(n-1)!_1 & ,\ n \in \mathbb{Z}^+ \\ \dfrac{(n+1)!_1}{n+1} & ,\ n \in \mathbb{Z}^- \setminus \{-1\}, \end{cases} \qquad \text{(Eq. 1.5.4)}$$

where

$$0!_1 = (-1)!_1 = 1\,. \qquad \text{(Eq. 1.5.5)}$$

Similarly, the recursive definition for the factorial of order $m = 2$ is given as:

$$n!_2 = \begin{cases} n(n-2)!_2 & ,\ n \in \mathbb{Z}^+ \\ \dfrac{(n+2)!_2}{n+2} & ,\ n \in \mathbb{Z}^- \setminus \{-1, -2\}, \end{cases} \qquad \text{(Eq. 3.5.4)}$$

where

$$0!_2 = (-1)!_2 = (-2)!_2 = 1\,. \qquad \text{(Eq. 3.5.5)}$$

From these, we can speculate on the recursive definition for $m = 3$, which would be as follows:

$$n!_3 = \begin{cases} n(n-3)!_3 & ,\ n \in \mathbb{Z}^+ \\ \dfrac{(n+3)!_3}{n+3} & ,\ n \in \mathbb{Z}^- \setminus \{-1, -2, -3\}, \end{cases} \qquad \text{(Eq. 6.5.1)}$$

where

$$0!_3 = (-1)!_3 = (-2)!_3 = (-3)!_3 = 1\,. \qquad \text{(Eq. 6.5.2)}$$

To derive this, we replaced instances of 2 with 3, added the additional seed $(-3)!_3 = 1$, and excluded -3 from the domain of the second case.

To verify that these seeds are sufficient to define the factorial of order 3 recursively, consider the following illustration:

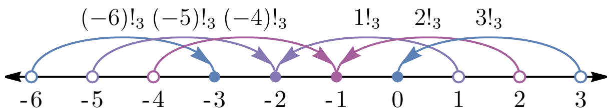

Fig. 6.5.1: $n!_3$ recursiveness

The figure demonstrates that the seeds -3 to 0 are indeed enough to define the factorial of order 3 recursively, as we have previously seen on the case of $n!_{(3)}$ in Subsection 5.2.

This is intuitive since, typically, three seeds are needed to define a recursive relationship for inputs spaced apart by 3. However, the pair -3 and 0 cannot be linked through the triple factorial's recursion due to division by 0. Consequently, we set these values as fixed starting points, stating that the recursive cases in Eq. 6.5.1 do not apply to them.

With the case of $m = 3$ examined, we are prepared to generalize. The simplicity of the expressions allows us to move forward without further detailed analysis.

To generalize, we need a compact expression to describe the seeds and a similarly concise domain for the second case.

To properly set all the required seeds, we can apply the following expression:

$$n!_m = 1\,, \quad \{n \in \mathbb{Z} \,|\, -m \leq n \leq 0\}\,. \qquad \text{(Eq. 6.5.3)}$$

Now let's focus on the domain of the second case. For $m = 1$, it excluded -1, as indicated by the set $n \in \mathbb{Z}^- \setminus \{-1\}$. For $m = 2$, the corresponding domain excluded -1 and -2. For $m = 3$, the value -3 was excluded as well.

This can be generalized as the domain of all negative integers excluding the first m numbers, written as:

$$\{n \in \mathbb{Z} \,|\, n < -m\}\,. \qquad \text{(Eq. 6.5.4)}$$

This equation selects only those integers less than $-m$, aligning with the definitions for $m = 1$ through $m = 3$.

By replacing instances of 3 in Eq. 6.5.1 with m and adjusting the seeds and domains accordingly, we arrive at the general recursive definition of the omnifactorial:

$$n!_m = \begin{cases} n(n-m)!_m & , \; n \in \mathbb{Z}^+ \\ \dfrac{(n+m)!_m}{n+m} & , \; \{n \in \mathbb{Z} \,|\, n < -m\} \, , \end{cases} \qquad \text{(Eq. 6.5.5)}$$

where

$$n!_m = 1 \, , \quad \{n \in \mathbb{Z} \,|\, -m \leq n \leq 0\} \, . \qquad \text{(Eq. 6.5.3)}$$

The two cases of this definition will be unified in Section 7, similar to how the cases for $m = 1$ were unified in Part 1 and for $m = 2$ in Section 4.

6.6 Falling product definition

In this subsection, we will derive the falling product definition for the omnifactorial. Instead of starting from scratch, we will build upon the falling product definitions for factorials of orders 1 and 2.

We begin with the known definition for $m = 1$:

$$n!_1 = n \cdot \prod_{k=0}^{-n-1} \frac{1}{n+k} \, , \quad n \in \mathbb{Z}^- . \qquad \text{(Eq. 3.6.1)}$$

For $m = 2$, the definition is:

$$n!_2 = n \cdot \prod_{k=0}^{\left\lceil \frac{-n}{2} \right\rceil - 1} \frac{1}{n+2k} \, , \quad n \in \mathbb{Z}^- . \qquad \text{(Eq. 3.6.2)}$$

To generalize this for any order m, we need to adjust two components: the upper limit of the $\prod$-product and the expression inside it.

The latter adjustment is straightforward because the term $(n+mk)^{-1}$ effectively represents the factors in a falling product for this context. In essence, $n!_m$ can be expressed as a product of negative reciprocals of every m-th integer, in line with the factorial order.

From Subsection 5.3 we know that the upper limit of the falling product of $n!_{(m)}$ is $\left\lceil \frac{n}{m} \right\rceil - 1$. This same form, with a negative sign in front of n, applies to the falling product definition of negative factorials of order 2.

This is accurate because the expression reflects the total number of factors in the product, which is $\left\lceil \frac{n}{m} \right\rceil$ when counting from 1. The behavior of the omnifactorial discussed in Subsections 6.2 and 6.3 confirms this pattern.

Thus, we generalize the falling product definition for the omnifactorial by replacing instances of 2 in Eq. 3.6.2 with m:

$$n!_m = n \cdot \prod_{k=0}^{\left\lceil \frac{-n}{m} \right\rceil - 1} \frac{1}{n+mk} \, , \quad n \in \mathbb{Z}^- . \qquad \text{(Eq. 6.6.1)}$$

Let's verify this result by testing it for $m = 3$. Here are three examples that represent every subset of integers when divided by 3:

$$(-6)!_3 = (-6) \cdot \prod_{k=0}^{\left\lceil \frac{6}{3} \right\rceil - 1} \frac{1}{-6+3k}$$

$$= (-6) \cdot \prod_{k=0}^{1} \frac{1}{-6+3k} = (-6) \cdot \frac{1}{-6} \cdot \frac{1}{-3} = -\frac{1}{3} \, ,$$

$$(-7)!_3 = (-7) \cdot \prod_{k=0}^{\left\lceil \frac{7}{3} \right\rceil - 1} \frac{1}{-7+3k}$$

$$= (-7) \cdot \prod_{k=0}^{2} \frac{1}{-7+3k} = (-7) \cdot \frac{1}{-7} \cdot \frac{1}{-4} \cdot \frac{1}{-1} = \frac{1}{4} \, ,$$

$$(-8)!_3 = (-8) \cdot \prod_{k=0}^{\left\lceil \frac{8}{3} \right\rceil - 1} \frac{1}{-8+3k}$$

$$= (-8) \cdot \prod_{k=0}^{2} \frac{1}{-8+3k} = (-8) \cdot \frac{1}{-8} \cdot \frac{1}{-5} \cdot \frac{1}{-2} = \frac{1}{10} \, .$$

These calculations confirm that Eq. 6.6.1 works well for $m = 3$. Testing for $m = 4$ provides further validation:

$$(-9)!_4 = (-9) \cdot \prod_{k=0}^{\left\lceil \frac{9}{4} \right\rceil - 1} \frac{1}{-9+4k}$$

$$= (-9) \cdot \prod_{k=0}^{2} \frac{1}{-9+4k} = (-9) \cdot \frac{1}{-9} \cdot \frac{1}{-5} \cdot \frac{1}{-1} = \frac{1}{5} \, .$$

You are encouraged to further test Eq. 6.6.1 with additional values or factorial orders, but its simplicity suggests that it will generally hold true without extensive verification.

This completes our examination of the falling product definition. Next, we will explore the rising product definition, followed by a summary of the results and conclusion of Section 6.

6.7 Rising product definition

In this subsection, we derive the rising product definition for the omnifactorial. We will build upon the definitions for rising products in factorials of orders 1 and 2, using out understanding to generalize to the omnifactorial case.

We start by recalling the definition for $m = 2$:

$$n!_2 = n \cdot \prod_{k=1}^{\left\lceil \frac{-n}{2} \right\rceil} \frac{1}{-2k + n \bmod 2} \, , \quad n \in \mathbb{Z}^- . \qquad \text{(Eq. 3.7.3)}$$

Let's also remember the multifactorial as a rising product:

$$n!_{(m)} = \prod_{k=1}^{\lceil \frac{n}{m} \rceil} \big(mk - (-n) \bmod m\big)\,, \quad n \in \mathbb{Z}^+.$$
(Eq. 5.4.6)

We can immediately generalize, since we now understand a few key concepts deeply and can build from what we have.

To begin with, we first address the upper limit of the $\prod$-product. For the omnifactorial, this limit will be the expression $\lceil \frac{-n}{m} \rceil$, because it correctly reflects the number of factors in the final product, consistent with the multifactorial case.

Next, the factors of the product are the reciprocals of integers, specifically of the form $\big(mk - (-n) \bmod m\big)$. This expression is used in the rising product definition of the multifactorial, and it only changes the order of the factors. For the omnifactorial, the terms should be the negative reciprocals of the expression $\big(mk - (-n) \bmod m\big)$.

However, when dealing with negative integers, $(-n) \bmod m$ becomes $n \bmod m$. This discrepancy arises because $(-n) \bmod m$ is different from $n \bmod m$, and directly substituting $-n$ could lead to incorrect results.

To align the rising product terms for negative integers with those of positive integers, we must adjust the expression to use n in place of $-n$. This adjustment ensures the modular arithmetic works as intended for negative integers.

Let's consider two examples for clarity. First, $11!_4$ is the product $3 \cdot 7 \cdot 11$, or in other words, the product of the terms $4k-1$, where k ranges from 1 to 3.

Second, $(-11)!_4$ is equal to $\frac{1}{-3} \cdot \frac{1}{-7}$ because negative factorials do not include n in their product expansion. Importantly, it involves terms of the form $-(4k-1)$, confirming consistency in the pattern when adjusted for negative integers.

The expression $4k-1$ is the result of $(-n) \bmod m$, when n is a positive number. In order for it to be valid for negative integers, we need to change $-n$ to n. Now, the expression behaves as intended.

Thus, the rising product definition for the omnifactorial is:

$$n!_m = n \cdot \prod_{k=1}^{\lceil \frac{-n}{m} \rceil} \frac{1}{-mk + n \bmod m}\,, \quad n \in \mathbb{Z}^-.$$
(Eq. 6.7.1)

Indeed, it works perfectly. To verify, let's calculate an example:

$$(-7)!_4 = (-7) \cdot \prod_{k=1}^{\lceil \frac{7}{4} \rceil} \frac{1}{-4k + (-7) \bmod 4}$$

$$= (-7) \cdot \prod_{k=1}^{2} \frac{1}{-4k+1} = (-7) \cdot \frac{1}{-3} \cdot \frac{1}{-7} = -\frac{1}{3}\,.$$

This result confirms the correctness of Eq. 6.7.1 and completes Section 6.

In conclusion, we have derived the rising product definition of the omnifactorial. We will now summarize the results and prepare for further generalizations in Section 7.

6.8 Synopsis

In short, Section 6 has explored the expansion of the multifactorial to include negative integers and sought to establish various definitions that describe it. We have developed 4 definitions in total, and we list them below.

The recursive definition of the omnifactorial is given by:

$$n!_m = \begin{cases} n(n-m)!_m & , \ n \in \mathbb{Z}^+ \\ \dfrac{(n+m)!_m}{n+m} & , \ \{n \in \mathbb{Z} \,|\, n < -m\}\,, \end{cases}$$
(Eq. 6.5.5)

where

$$n!_m = 1\,, \quad \{n \in \mathbb{Z} \,|\, -m \le n \le 0\}\,. \quad \text{(Eq. 6.5.3)}$$

The Roman-like definition of the omnifactorial is as follows:

$$\lfloor n \rceil!_m = \begin{cases} n!_{(m)} & , \ n \in \mathbb{Z}_0^+ \\ \dfrac{(-1)^{\frac{n - n \bmod m}{m}+1}}{(-n-m)!_{(m)}} & , \ n \in \mathbb{Z}^-, \end{cases}$$
(Eq. 6.4.2)

where the multifactorial is defined recursively as:

$$n!_{(m)} = n(n-m)!_{(m)}\,, \quad n \in \mathbb{Z}^+, \quad \text{(Eq. 5.2.6)}$$

with seeds given by:

$$n!_{(m)} = 1\,, \quad \{n \in \mathbb{Z} \,|\, -m < n \le 0\}\,. \quad \text{(Eq. 5.2.7)}$$

The falling product of the omnifactorial is of the form:

$$n!_m = n \cdot \prod_{k=0}^{\lceil \frac{-n}{m} \rceil - 1} \frac{1}{n+mk}\,, \quad n \in \mathbb{Z}^-. \quad \text{(Eq. 6.6.1)}$$

Finally, the rising product definition of $n!_m$ was found to be:

$$n!_m = n \cdot \prod_{k=1}^{\lceil \frac{-n}{m} \rceil} \frac{1}{-mk + n \bmod m}\,, \quad n \in \mathbb{Z}^-.$$
(Eq. 6.7.1)

In the next section, we will unite these relationships with their positive counterparts, and reach universal definitions that describe the omnifactorial across all integers in a single expression.

These definitions collectively offer a comprehensive framework for understanding the omnifactorial across all integers. In the upcoming section, we will integrate these definitions with their positive counterparts to formulate unified expressions that encapsulate $n!_m$ across all integers in universal definitions.

7 Omnifactorial generalizations

7.1 Introduction

In this section, we will consolidate the piece-wise definitions of the omnifactorial introduced earlier into unified forms. The *generalization process* will follow a similar approach to what was used for factorials of orders 1 and 2.

We will start by combining the two cases of the recursive definition, followed by unifying the Roman-like definition of the omnifactorial. Next, we will address the two $\prod$-product definitions in a similar manner. Finally, we will present the generalized results and bring this paper to a conclusion.

7.2 Recursive definition step 1: $\theta(n)$

In this subsection, we will unify the two cases of the recursive definition of the omnifactorial. This generalization closely mirrors the process followed in Section 4, but we will revisit the steps here.

Let's start by recalling the recursive definition:

$$n!_m = \begin{cases} n(n-m)!_m & , \ n \in \mathbb{Z}^+ \\ \dfrac{(n+m)!_m}{n+m} & , \ \{n \in \mathbb{Z} \mid n < -m\}\,, \end{cases} \quad \text{(Eq. 6.5.5)}$$

where

$$n!_m = 1\,, \quad \{n \in \mathbb{Z} \mid -m \le n \le 0\}\,. \quad \text{(Eq. 6.5.3)}$$

To aid in this generalization, we will introduce abbreviations for the domains of our piece-wise definitions. Specifically, for the recursive definition of the omnifactorial, we define the sets $\mathbb{S}_1 = \mathbb{Z}^+$ and $\mathbb{S}_2 = \{n \in \mathbb{Z}^- \mid n < -m\}$.

Using these abbreviations, we can rewrite the expression as:

$$n!_m = \begin{cases} n(n-m)!_m & , \ n \in \mathbb{S}_1 \\ \dfrac{(n+m)!_m}{n+m} & , \ n \in \mathbb{S}_2\,. \end{cases} \quad \text{(Eq. 6.5.5)}$$

Now, let's proceed with the first step of the *generalization process*. As seen in Subsection 4.2, we can rewrite the cases as follows:

$$n!_m = \begin{cases} n^{+1} \cdot \big(n - m(+1)\big)!_m & , \ n \in \mathbb{S}_1 \\ (n+m)^{-1} \cdot \big(n - m(-1)\big)!_m & , \ n \in \mathbb{S}_2\,. \end{cases}$$

This highlights the points where the *F.F.* $\theta(n)$ applies, defined as:

$$\theta(n) = \frac{\delta(n)}{|\delta(n)|} = \begin{cases} 1 & , \ n \in \mathbb{S}_1 \\ -1 & , \ n \in \mathbb{S}_2\,. \end{cases} \quad \text{(Eq. 1.4.2)}$$

Incorporating $\theta(n)$ into the definition gives us:

$$n!_m = \begin{cases} n^{\theta(n)}\big(n - m\,\theta(n)\big)!_m & , \ n \in \mathbb{S}_1 \\ (n+m)^{\theta(n)}\big(n - m\,\theta(n)\big)!_m & , \ n \in \mathbb{S}_2\,. \end{cases} \quad \text{(Eq. 7.2.1)}$$

7.3 Recursive definition step 2: $\xi'(n)$

At this point, the only difference between the two cases lies in the left-most term. In the first case, it's $n^{\theta(n)}$, while in the second, it's $(n+m)^{\theta(n)}$. This difference is shown below:

$$n!! = \begin{cases} (n+0)^{\theta(n)}\big(n - m\,\theta(n)\big)!_m & , \ n \in \mathbb{S}_1 \\ (n+m)^{\theta(n)}\big(n - m\,\theta(n)\big)!_m & , \ n \in \mathbb{S}_2\,. \end{cases} \quad \text{(Eq. 7.3.1)}$$

The term that varies between 0 and m can be generalized as $m\,\xi'(n)$, defined as:

$$m\,\xi'(n) = \frac{m - m\,\theta(n)}{2} = \begin{cases} 0 & , \ n \in \mathbb{S}_1 \\ m & , \ n \in \mathbb{S}_2\,. \end{cases} \quad \text{(Eq. 1.4.4)}$$

Substituting this into Eq. 7.2.1, we obtain:

$$\begin{cases} \big(n + m\,\xi'(n)\big)^{\theta(n)}\big(n - m\,\theta(n)\big)!_m & , \ n \in \mathbb{S}_1 \\ \big(n + m\,\xi'(n)\big)^{\theta(n)}\big(n - m\,\theta(n)\big)!_m & , \ n \in \mathbb{S}_2\,. \end{cases} \quad \text{(Eq. 7.3.2)}$$

Therefore, the two cases can be merged into a single expression. The generalized recursive definition of the omnifactorial for all integers is defined as:

$$n!_m = \big(n + m\,\xi'(n)\big)^{\theta(n)}\big(n - m\,\theta(n)\big)!_m\,, \quad n \in \mathbb{D}_2\,, \quad \text{(Eq. 7.3.3)}$$

where

$$\mathbb{D}_2 = \{n \in \mathbb{Z} \mid n < -m \text{ or } n > 0\}\,, \quad \text{(Eq. 7.3.4)}$$

and

$$n!_m = 1\,, \quad \{n \in \mathbb{Z} \mid -m \le n \le 0\}\,. \quad \text{(Eq. 6.5.3)}$$

The set $\mathbb{D}_2$ represents the domain in which the generalized recursive definition of the omnifactorial is valid.

7.4 Roman-like definition step 1: $\eta(n, m)$

In the following subsections, we will merge the piece-wise definition of $n!_m$ that mirrors the Roman factorial. The Roman-like piece-wise definition of the omnifactorial is presented as:

$$\lfloor n \rceil!_m = \begin{cases} n!_{(m)} & , \ n \in \mathbb{Z}_0^+ \\ \dfrac{(-1)^{\frac{n - n \bmod m}{m} + 1}}{(-n-m)!_{(m)}} & , \ n \in \mathbb{Z}^-, \end{cases} \quad \text{(Eq. 6.4.2)}$$

in which the multifactorial is defined recursively as

$$n!_{(m)} = n(n-m)!_{(m)}\,, \quad n \in \mathbb{Z}^+, \quad \text{(Eq. 5.2.6)}$$

with

$$n!_{(m)} = 1\,, \quad \{n \in \mathbb{Z} \mid -m < n \le 0\}\,. \quad \text{(Eq. 5.2.7)}$$

In Subsection 4.4, the first move was to introduce the function $\eta(n, 2)$ to substitute the numerator in the second case of Eq. 3.4.8. Now, we need to define $\eta(n,m)$ in such a way to fit within our current generalization.

Despite the complex exponent of (-1) in the second case of Eq. 6.4.2, the base simplifies to $(+1)$ for positive integers, making it equal to 1 for that domain and leaving the first case unaffected.

Therefore, we define $\eta(n,m)$ as:

$$\eta(n,m) = \theta(n)^{\frac{n-n \bmod m}{m}+1}, \quad n \in \mathbb{R}. \quad \text{(Eq. 7.4.1)}$$

While unnecessary for this paper, in the previous one, we extended *F.F.* to apply to all real numbers. We choose to continue that practice by checking if the exponent of $\eta(n,m)$ is always an integer, preventing complex outputs. Luckily, $\frac{n-n \bmod m}{m}$ is already an integer, even for any real n. This is detailed in Addendum 10.6.

Proceeding with the generalization, substituting $\eta(m,n)$ into Eq. 6.4.2 gives:

$$\lfloor n \rceil !_m = \begin{cases} \eta(n,m) \cdot n!_{(m)} & , \ n \in \mathbb{Z}_0^+ \\ \dfrac{\eta(n,m)}{(-n-m)!_{(m)}} & , \ n \in \mathbb{Z}^- . \end{cases} \quad \text{(Eq. 7.4.2)}$$

7.5 Roman-like definition step 2: $\theta(n)$

From this point, the generalization follows the same steps as in Subsections 4.5 and 4.6. Our next task is to emphasize the exponent of the omnifactorial to introduce another *F.F.*:

$$\lfloor n \rceil !_m = \begin{cases} \eta(n,m) \cdot \left(n!_{(m)}\right)^1 & , \ n \in \mathbb{Z}_0^+ \\ \eta(n,m) \cdot \left[(-n-m)!_{(m)}\right]^{-1} & , \ n \in \mathbb{Z}^- . \end{cases} \quad \text{(Eq. 7.5.1)}$$

This is another instance where $\theta(n)$ is used, defined as:

$$\theta(n) = \frac{\delta(n)}{|\delta(n)|} = \begin{cases} 1 & , \ n \in \mathbb{Z}_0^+ \\ -1 & , \ n \in \mathbb{Z}^- . \end{cases} \quad \text{(Eq. 1.4.2)}$$

We can now incorporate $\theta(n)$ into Eq. 7.5.1 to obtain:

$$\lfloor n \rceil !_m = \begin{cases} \eta(n,m) \cdot \left(n!_{(m)}\right)^{\theta(n)} & , \ n \in \mathbb{Z}_0^+ \\ \eta(n,m) \cdot \left[(-n-m)!_{(m)}\right]^{\theta(n)} & , \ n \in \mathbb{Z}^- . \end{cases} \quad \text{(Eq. 7.5.2)}$$

Next, let's clear up the expressions. We will use $|n|$ to simplify them further:

$$\lfloor n \rceil !_m = \begin{cases} \eta(n,m) \cdot |n|!_{(m)}^{\theta(n)} & , \ n \in \mathbb{Z}_0^+ \\ \eta(n,m) \cdot \left(|n|-m\right)!_{(m)}^{\theta(n)} & , \ n \in \mathbb{Z}^- . \end{cases} \quad \text{(Eq. 7.5.3)}$$

7.6 Roman-like definition step 3: $\xi'(n)$

For the final step, note the subtraction of m from $|n|$ in the second case, which is not present in the first. This distinction is highlighted below:

$$\lfloor n \rceil !_m = \begin{cases} \eta(n,m) \cdot \left(|n|-0\right)!_{(m)}^{\theta(n)} & , \ n \in \mathbb{Z}_0^+ \\ \eta(n,m) \cdot \left(|n|-m\right)!_{(m)}^{\theta(n)} & , \ n \in \mathbb{Z}^- . \end{cases} \quad \text{(Eq. 7.6.1)}$$

This behavior matches how $\xi'(n)$ operates when multiplied by m:

$$m\,\xi'(n) = \frac{m - m\,\theta(n)}{2} = \begin{cases} 0 & , \ n \in \mathbb{Z}_0^+ \\ m & , \ n \in \mathbb{Z}^- . \end{cases} \quad \text{(Eq. 1.4.4)}$$

Substituting this expression unifies the two cases. The result is:

$$\lfloor n \rceil !_m = \begin{cases} \eta(n,m) \cdot \left(|n|-m\,\xi'(n)\right)!_{(m)}^{\theta(n)} & , \ n \in \mathbb{Z}_0^+ \\ \eta(n,m) \cdot \left(|n|-m\,\xi'(n)\right)!_{(m)}^{\theta(n)} & , \ n \in \mathbb{Z}^- . \end{cases} \quad \text{(Eq. 7.6.2)}$$

Thus, the generalized Roman-like definition of the omnifactorial is as follows:

$$\lfloor n \rceil !_m = \eta(n,m) \cdot \left(|n|-m\,\xi'(n)\right)!_{(m)}^{\theta(n)}, \ n \in \mathbb{Z}, \quad \text{(Eq. 7.6.3)}$$

where the multifactorial is defined as

$$n!_{(m)} = n(n-m)!_{(m)}, \quad n \in \mathbb{Z}^+, \quad \text{(Eq. 5.2.6)}$$

with

$$n!_{(m)} = 1, \quad \{n \in \mathbb{Z} \mid -m < n \le 0\}. \quad \text{(Eq. 5.2.7)}$$

7.7 $\prod$-product definitions step 1: $|n|$

In the upcoming subsections, we will generalize the two $\prod$-product definitions of the omnifactorial. They are shown in the form of tables below:

$n!_m$	Falling product
$n \in \mathbb{Z}_0^+$	$\displaystyle\prod_{k=0}^{\lceil \frac{n}{m} \rceil -1} (n-mk)$
$n \in \mathbb{Z}^-$	$\displaystyle n \cdot \prod_{k=0}^{\lceil \frac{-n}{m} \rceil -1} \frac{1}{n+mk}$

Tbl. 7.7.1: $n!_m$ as a falling product

$n!_m$	Rising product
$n \in \mathbb{Z}_0^+$	$\displaystyle\prod_{k=1}^{\lceil \frac{n}{m} \rceil} \left(mk - (-n) \bmod m\right)$
$n \in \mathbb{Z}^-$	$\displaystyle n \cdot \prod_{k=1}^{\lceil \frac{-n}{m} \rceil} \frac{1}{-mk + n \bmod m}$

Tbl. 7.7.2: $n!_m$ as a rising product

Our first step in the *generalization process* is to introduce the absolute value of n, denoted by $|n|$, into these definitions. This addition is straightforward for the falling product. However, in the case of the rising product, we need to account for the quantity $(-n) \bmod m$. Interestingly, this is equivalent to $n \bmod m$ when n is a negative integer, which necessitates substituting $-|n|$ instead of simply $|n|$.

The definitions can be rewritten as follows:

$n!_m$	Falling product
$n \in \mathbb{Z}_0^+$	$\displaystyle\prod_{k=0}^{\lceil \frac{\|n\|}{m} \rceil - 1} (n - mk)$
$n \in \mathbb{Z}^-$	$\displaystyle n \cdot \prod_{k=0}^{\lceil \frac{\|n\|}{m} \rceil - 1} \frac{1}{n + mk}$

Tbl. 7.7.3: $n!_m$ as a falling product (generalization step 1)

$n!_m$	Rising product
$n \in \mathbb{Z}_0^+$	$\displaystyle\prod_{k=1}^{\lceil \frac{\|n\|}{m} \rceil} \Big(mk - \big(-\|n\|\big) \bmod m\Big)$
$n \in \mathbb{Z}^-$	$\displaystyle n \cdot \prod_{k=1}^{\lceil \frac{\|n\|}{m} \rceil} \frac{1}{-mk + \big(-\|n\|\big) \bmod m}$

Tbl. 7.7.4: $n!_m$ as a rising product (generalization step 1)

Although these generalized expressions might initially seem complex, it is essential to emphasize that the *generalization process* follows a logical progression. The introduction of $|n|$ and $-|n|$ is a necessary refinement to ensure consistency across all values of n. This approach mirrors the generalization we performed for the factorial of order 2, which is itself similar to its first application, for $m = 1$.

Regardless the similarities however, in this study it is considered important to explain everything in detail and not to skip anything that could be considered trivial, as this is where mistakes and oversights usually happen.

7.8 $\prod$-product definitions step 2: $\Phi(n)$

Our next objective is to handle the term n that lies outside of the $\prod$-products. As we've seen before in Subsection 4.8, the function $\Phi(n)$ was made exactly for this purpose:

$$\Phi(n) = \big(n + \Theta(n)\big)^{\xi'(n)} = \begin{cases} 1\,, & n \in \mathbb{Z}_0^+ \\ n\,, & n \in \mathbb{Z}^-. \end{cases} \qquad \text{(Eq. 1.4.10)}$$

By incorporating $\Phi(n)$ into our definitions, they now appear as follows:

$n!_m$	Falling product
$n \in \mathbb{Z}_0^+$	$\displaystyle \Phi(n) \cdot \prod_{k=0}^{\lceil \frac{\|n\|}{m} \rceil - 1} (n - mk)$
$n \in \mathbb{Z}^-$	$\displaystyle \Phi(n) \cdot \prod_{k=0}^{\lceil \frac{\|n\|}{m} \rceil - 1} \frac{1}{n + mk}$

Tbl. 7.8.1: $n!_m$ as a falling product (generalization step 2)

$n!_m$	Rising product
$n \in \mathbb{Z}_0^+$	$\displaystyle \Phi(n) \cdot \prod_{k=1}^{\lceil \frac{\|n\|}{m} \rceil} \Big(mk - \big(-\|n\|\big) \bmod m\Big)$
$n \in \mathbb{Z}^-$	$\displaystyle \Phi(n) \cdot \prod_{k=1}^{\lceil \frac{\|n\|}{m} \rceil} \frac{1}{-mk + \big(-\|n\|\big) \bmod m}$

Tbl. 7.8.2: $n!_m$ as a rising product (generalization step 2)

7.9 $\prod$-product definitions step 3: $\theta(n)$

The $\prod$-product definitions differ between cases only through their index terms. To clarify, we present these terms in the tables below:

Indices	Falling product
$n \in \mathbb{Z}_0^+$	$(n - mk)$
$n \in \mathbb{Z}^-$	$\dfrac{1}{n + mk}$

Tbl. 7.9.1: Index terms in the falling product definitions of the omnifactorial

Indices	Rising product
$n \in \mathbb{Z}_0^+$	$\Big(mk - \big(-\|n\|\big) \bmod m\Big)$
$n \in \mathbb{Z}^-$	$\dfrac{1}{-mk + \big(-\|n\|\big) \bmod m}$

Tbl. 7.9.2: Index terms in the rising product definitions of the omnifactorial

The cases of falling product are reminiscent of the function $\theta(n)$. Specifically, the whole expression is raised to the power of $\theta(n)$, and the term mk is multiplied by it as well.

Applying this modification to Tbl. 7.9.1 leads to the following updated expression:

Index	Falling product
$n \in \mathbb{Z}$	$\big(n - mk\,\theta(n)\big)^{\theta(n)}$

Tbl. 7.9.3: Index term in the falling product definition of the omnifactorial, generalized

Similarly, for the rising product, the modification involves raising the expression to the power of $\theta(n)$ and multiplying it by the same function (the order of these operations is irrelevant). The updated expression is:

Index	Rising product
$n \in \mathbb{Z}$	$\Big[\theta(n)\Big(mk - \big(-\lvert n\rvert\big) \bmod m\Big)\Big]^{\theta(n)}$

Tbl. 7.9.4: Index term in the rising product definition of the omnifactorial, generalized

Therefore, the $\prod$-products are now unified into two universal definitions.

The falling product definition of the omnifactorial is hereby defined as follows:

$$n!_m = \Phi(n) \cdot \prod_{k=0}^{\left\lceil \frac{|n|}{m} \right\rceil - 1} \big(n - mk\,\theta(n)\big)^{\theta(n)}\,, \quad n \in \mathbb{Z}\,. \tag{Eq. 7.9.1}$$

Similarly, the rising product definition of $n!_m$ for $n \in \mathbb{Z}$ is:

$$n!_m = \Phi(n) \cdot \prod_{k=1}^{\left\lceil \frac{|n|}{m} \right\rceil} \Big[\theta(n)\Big(mk - \big(-|n|\big) \bmod m\Big)\Big]^{\theta(n)} \tag{Eq. 7.9.2}$$

These results conclude Section 7. A summary follows, leading into conclusions and wrapping up the second part of the study.

7.10 Synopsis

In this section, we have generalized the piece-wise definitions of the omnifactorial into single, universal expressions by applying the *generalization process* through a few *foundational functions.*

The generalized recursive definition of $n!_m$ is defined as:

$$n!_m = \big(n + m\,\xi'(n)\big)^{\theta(n)}\big(n - m\,\theta(n)\big)!_m\,, \quad n \in \mathbb{D}_2\,, \tag{Eq. 7.3.3}$$

where

$$\mathbb{D}_2 = \{n \in \mathbb{Z} \,|\, n < -m \text{ or } n > 0\}\,, \tag{Eq. 7.3.4}$$

and

$$n!_m = 1\,, \quad \{n \in \mathbb{Z} \,|\, -m \le n \le 0\}\,. \tag{Eq. 6.5.3}$$

The generalized Roman-like definition of the omnifactorial is as follows:

$$\lfloor n \rceil !_m = \eta(n,m) \cdot \big(|n| - m\,\xi'(n)\big)!_{(m)}^{\theta(n)}\,, \quad n \in \mathbb{Z}\,, \tag{Eq. 7.6.3}$$

in which the multifactorial is defined as

$$n!_{(m)} = n(n-m)!_{(m)}\,, \quad n \in \mathbb{Z}^+, \tag{Eq. 5.2.6}$$

where

$$n!_{(m)} = 1\,, \quad \{n \in \mathbb{Z} \,|\, -m < n \le 0\}\,. \tag{Eq. 5.2.7}$$

The falling product definition of the omnifactorial for $n \in \mathbb{Z}$ is defined as follows:

$$n!_m = \Phi(n) \cdot \prod_{k=0}^{\left\lceil \frac{|n|}{m} \right\rceil - 1} \big(n - mk\,\theta(n)\big)^{\theta(n)}\,, \quad n \in \mathbb{Z}\,. \tag{Eq. 7.9.1}$$

Lastly, the rising product definition of $n!_m$ for all $n \in \mathbb{Z}$ is defined as:

$$n!_m = \Phi(n) \cdot \prod_{k=1}^{\left\lceil \frac{|n|}{m} \right\rceil} \Big[\theta(n)\Big(mk - \big(-|n|\big) \bmod m\Big)\Big]^{\theta(n)}\,. \tag{Eq. 7.9.2}$$

These results bring Section 7 to a close, and with it, the conclusion of this paper.

8 Conclusions

8.1 Summary

In short:

- **Section 2**: We introduced the double factorial and found various recursive and non-recursive definitions that describe it.
- **Section 3**: We extended the double factorial into negative integers, naming it *factorial of order* 2 and found 4 definitions that express it.
- **Section 4**: We applied the *generalization process* to unify the piece-wise definitions of the factorial of order 2.
- **Section 5**: We introduced the multifactorial and built a few definitions, in a similar manner to Section 2.
- **Section 6**: We extended the multifactorial into negative integers, assigning it the term *omnifactorial* and developed various definitions.
- **Section 7**: We applied the *generalization process* to unite the cases of the piece-wise omnifactorial definitions, resulting in 4 universal expressions.

Also, In Subsection 7.4 we defined the following *foundational function*:

$$\eta(n,m) = \theta(n)^{\frac{n-n \bmod m}{m}+1}, \quad n \in \mathbb{R}. \quad \text{(Eq. 7.4.1)}$$

8.2 Results

The results of this paper are laid out in two pages. This page contains the definitions of $n!_2$, while the next page lists the definitions of $n!_m$.

In Section 4 we consolidated various definitions of the factorial of order 2. The piece-wise definition of $n!_2$ was expressed as:

$$n!_2 = \begin{cases} n(n-2)!_2 & , \ n \in \mathbb{Z}^+ \\ \dfrac{(n+2)!_2}{n+2} & , \ n \in \mathbb{Z}^- \setminus \{-1, -2\}, \end{cases} \quad \text{(Eq. 3.5.4)}$$

where

$$0!_2 = (-1)!_2 = (-2)!_2 = 1 . \quad \text{(Eq. 3.5.5)}$$

The generalized version is as follows:

$$n!_2 = \big(n + 2\,\xi'(n)\big)^{\theta(n)} \big(n - 2\,\theta(n)\big)!_2 , \quad n \in \mathbb{D}_1 , \quad \text{(Eq. 4.3.3)}$$

where

$$\mathbb{D}_1 = \{n \in \mathbb{Z} \,|\, n \neq 0, \ -1, \ -2\} , \quad \text{(Eq. 4.3.4)}$$

with

$$0!_2 = (-1)!_2 = (-2)!_2 = 1 . \quad \text{(Eq. 4.3.5)}$$

The Roman-like definition of the double factorial was initially expressed as:

$$\lfloor n \rceil !_2 = \begin{cases} n!_{(2)} & , \ n \in \mathbb{Z}_0^+ \\ \dfrac{(-1)^{\left\lceil \frac{n+1}{2} \right\rceil}}{(-n-2)!_{(2)}} & , \ n \in \mathbb{Z}^-, \end{cases} \quad \text{(Eq. 3.4.8)}$$

where

$$n!_{(2)} = n(n-2)!_{(2)} , \ \ 0!_{(2)} = (-1)!_{(2)} = 1 , \ \ n \in \mathbb{Z}^+ . \quad \text{(Eq. 3.4.3)}$$

Next, the generalized definition of $\lfloor n \rceil !_2$ is defined as follows:

$$\lfloor n \rceil !_2 = \eta(n,2) \cdot \big(|n| - 2\,\xi'(n)\big)!_{(2)}^{\theta(n)} , \ \ n \in \mathbb{Z} , \quad \text{(Eq. 4.6.3)}$$

where

$$n!_{(2)} = n(n-2)!_{(2)} , \ \ 0!_{(2)} = (-1)!_{(2)} = 1 , \ \ n \in \mathbb{Z}^+ . \quad \text{(Eq. 3.4.3)}$$

The piece-wise $\prod$-product definitions of the double factorial were found to be:

$n!_2$	Falling product
$n \in \mathbb{Z}^+$	$\displaystyle\prod_{k=0}^{\left\lceil \frac{n}{2} \right\rceil -1} (n-2k)$
$n \in \mathbb{Z}^-$	$\displaystyle n \cdot \prod_{k=0}^{\left\lceil \frac{-n}{2} \right\rceil -1} \frac{1}{n+2k}$

Tbl. 4.7.1: $n!_2$ as a falling product

$n!_2$	Rising product
$n \in \mathbb{Z}^+$	$\displaystyle\prod_{k=1}^{\left\lceil \frac{n}{2} \right\rceil} (2k - n \bmod 2)$
$n \in \mathbb{Z}^-$	$\displaystyle n \cdot \prod_{k=1}^{\left\lceil \frac{-n}{2} \right\rceil} \frac{1}{-2k + n \bmod 2}$

Tbl. 4.7.2: $n!_2$ as a rising product

Finally, the generalized, non-recursive $\prod$-product definitions of $n!_2$ as a falling or as a rising product are defined as:

$$n!_2 = \Phi(n) \cdot \prod_{k=0}^{\left\lceil \frac{|n|}{2} \right\rceil -1} \big(n - 2k\,\theta(n)\big)^{\theta(n)} , \quad n \in \mathbb{Z} . \quad \text{(Eq. 4.9.1)}$$

$$n!_2 = \Phi(n) \cdot \prod_{k=1}^{\left\lceil \frac{|n|}{2} \right\rceil} \big(\theta(n) \cdot (2k - n \bmod 2)\big)^{\theta(n)} , \ n \in \mathbb{Z} . \quad \text{(Eq. 4.9.2)}$$

In Section 7 we did the same thing for the factorial of order m, also known as the omnifactorial. Its recursive definition was iterated as:

$$n!_m = \begin{cases} n(n-m)!_m & , \; n \in \mathbb{Z}^+ \\ \dfrac{(n+m)!_m}{n+m} & , \; \{n \in \mathbb{Z} \,|\, n < -m\}\,, \end{cases} \quad \text{(Eq. 6.5.5)}$$

where

$$n!_m = 1\,, \quad \{n \in \mathbb{Z} \,|\, -m \le n \le 0\}\,. \quad \text{(Eq. 6.5.3)}$$

The generalized recursive definition of the omnifactorial is defined as:

$$n!_m = \big(n + m\,\xi'(n)\big)^{\theta(n)} \big(n - m\,\theta(n)\big)!_m\,, \quad n \in \mathbb{D}_2\,, \quad \text{(Eq. 7.3.3)}$$

where

$$\mathbb{D}_2 = \{n \in \mathbb{Z} \,|\, n < -m \text{ or } n > 0\}\,, \quad \text{(Eq. 7.3.4)}$$

and

$$n!_m = 1\,, \quad \{n \in \mathbb{Z} \,|\, -m \le n \le 0\}\,. \quad \text{(Eq. 6.5.3)}$$

The piece-wise Roman-like definition of the omnifactorial is iterated as follows:

$$\lfloor n \rceil !_m = \begin{cases} n!_{(m)} & , \; n \in \mathbb{Z}_0^+ \\ \dfrac{(-1)^{\frac{n - n \bmod m}{m} + 1}}{(-n-m)!_{(m)}} & , \; n \in \mathbb{Z}^-, \end{cases} \quad \text{(Eq. 6.4.2)}$$

in which the multifactorial is defined recursively as

$$n!_{(m)} = n(n-m)!_{(m)}\,, \quad n \in \mathbb{Z}^+, \quad \text{(Eq. 5.2.6)}$$

where

$$n!_{(m)} = 1\,, \quad \{n \in \mathbb{Z} \,|\, -m < n \le 0\}\,. \quad \text{(Eq. 5.2.7)}$$

The generalized Roman-like definition of the omnifactorial is as follows:

$$\lfloor n \rceil !_m = \eta(n,m) \cdot \big(|n| - m\,\xi'(n)\big)!_{(m)}^{\theta(n)}\,, \quad n \in \mathbb{Z}\,, \quad \text{(Eq. 7.6.3)}$$

in which the multifactorial is defined as

$$n!_{(m)} = n(n-m)!_{(m)}\,, \quad n \in \mathbb{Z}^+, \quad \text{(Eq. 5.2.6)}$$

where

$$n!_{(m)} = 1\,, \quad \{n \in \mathbb{Z} \,|\, -m < n \le 0\}\,. \quad \text{(Eq. 5.2.7)}$$

The $\prod$-product definitions of $n!_m$ are shown in the form of tables below:

$n!_m$	Falling product
$n \in \mathbb{Z}_0^+$	$\displaystyle\prod_{k=0}^{\lceil \frac{n}{m} \rceil - 1} (n - mk)$
$n \in \mathbb{Z}^-$	$\displaystyle n \cdot \prod_{k=0}^{\lceil \frac{-n}{m} \rceil - 1} \frac{1}{n + mk}$

Tbl. 7.7.1: $n!_m$ as a falling product

$n!_m$	Rising product
$n \in \mathbb{Z}_0^+$	$\displaystyle\prod_{k=1}^{\lceil \frac{n}{m} \rceil} \big(mk - (-n) \bmod m\big)$
$n \in \mathbb{Z}^-$	$\displaystyle n \cdot \prod_{k=1}^{\lceil \frac{-n}{m} \rceil} \frac{1}{-mk + n \bmod m}$

Tbl. 7.7.2: $n!_m$ as a rising product

The generalized falling product definition of the omnifactorial is defined as follows:

$$n!_m = \Phi(n) \cdot \prod_{k=0}^{\lceil \frac{|n|}{m} \rceil - 1} \big(n - mk\,\theta(n)\big)^{\theta(n)}\,, \quad n \in \mathbb{Z}\,. \quad \text{(Eq. 7.9.1)}$$

Lastly, the generalized rising product definition of $n!_m$ for $n \in \mathbb{Z}$ is defined as:

$$n!_m = \Phi(n) \cdot \prod_{k=1}^{\lceil \frac{|n|}{m} \rceil} \Big[\theta(n)\Big(mk - \big(-|n|\big) \bmod m\Big)\Big]^{\theta(n)}\,. \quad \text{(Eq. 7.9.2)}$$

Note that the generalized definitions of the factorial of order 2 differ significantly from those of the omnifactorial. This is primarily due to the role of modular arithmetic, which simplifies considerably when $m = 2$.

For example, in defining the Roman factorial for orders 2 and m, we introduced the function $\eta(n,m)$ to aid in our generalization. Initially, this function relied on rounding up to the nearest integer, but we later adapted it to incorporate modular arithmetic.

These differences, while ultimately equivalent, highlight the necessity of re-generalizing from the base case each time we introduce a new factorial variation, rather than simply modifying an existing generalized definition that applies to a narrower domain.

In the next part, we will explore a continuation of the factorial into non-integers. We will initially examine the Gamma function, but ultimately choose an alternative way to define this variation of the factorial. We will create definitions for positive and negative numbers separately, then combine them into unified expressions using the *generalization process.* This approach will be extended to the double factorial and the multifactorial, leading to expanded definitions of the omnifactorial that encapsulate all real numbers.

9 Details and References

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Acknowledgements

Special thanks go to Padelis Venardos, a math teacher without whom I would not have the aspiration to begin this project, nor the will to finish it.

Author's Contributions

The author has conceived the ideas, made the calculations and has also written and approved the manuscript. This work was originally submitted to ResearchGate on August 21 (2024), the 235th birthday of Augustin-Louis Cauchy.

Software Used

This document was made in LaTeX using the online platform Overleaf, compiled in TeX Live 2023. All graphs were drawn in Mathematica. ChatGPT 3.5, ChatGPT 4o and Claude 3.5 Sonnet have assisted in the formatting of this document, as well as in some of the figures.

Code Availability

The Mathematica code used to generate the figures and perform the calculations in this document, as well as the LaTeX code, are available upon request.

For any questions, feedback, error correction or further discussion, feel free to contact me at this email: lliponis@physics.auth.gr.

10 Addendum

Some entries include information from Part 1, but have been slightly updated to match the needs of this paper.

10.1 Number sets

A set is a fundamental concept in mathematics, representing a collection of distinct elements or members [6] [7]. Certain sets are so essential in mathematics that they are given special names and notations, often denoted in blackboard bold typeface (e.g. $\mathbb{Z}$). Below are some of the most common ones:

Set	Description	Example
$\mathbb{N}\,,\mathbb{Z}^+$	Natural numbers	$1,\ 2,\ 3,\ 4\ \cdots$
$\mathbb{N}_0$	$\mathbb{N}$ with 0 (or $\mathbb{Z}_0^+$)	$0,\ 1,\ 2,\ 3\ \cdots$
$\mathbb{Z}$	Integers	$\cdots$ -2, -1, 0, 1 $\cdots$
$\mathbb{Q}$	Rational numbers	$^1/_2$, -$^5/_4$, 0.01 $\cdots$
$\mathbb{R}$	Real numbers	$\sqrt{2},\ \pi,\ e,\ \phi\ \cdots$
$\mathbb{I}$	Imaginary numbers	i, $9.7i$, -$^i/_{25}$ $\cdots$
$\mathbb{C}$	Complex numbers	$1+i,\ \sqrt{3}-6i\ \cdots$

Tbl. 10.1.1: Common number sets

It is important to note that the set $\mathbb{C}$ encompasses all real and imaginary numbers. For example, the numbers $2-3i$, $5.2i$, and 4 are all elements of the complex number set.

By combining these number sets, we can define other sets. For instance, the set of all real numbers excluding negative integers can be expressed as:

$$\mathbb{R} \setminus \mathbb{Z}^-\,,$$

where $\mathbb{Z}^-$ represents the set of negative integers.

Additionally, a number set denoted with an asterisk (*) typically excludes the number 0. For example, $\mathbb{Z}^*$ represents the set of all integers except 0.

When a set cannot be easily represented by a list of elements, there are two common notations for expressing it in detail. For example, the set of all positive odd integers except 1 can be written as:

$$\mathbb{Z}^+_{odd} \setminus \{1\}\,.$$

Alternatively, we can use set-builder notation, a method that concisely defines sets with more complex characteristics. For example, the set of all real numbers n greater than or equal to 3 is represented as:

$$\{n \in \mathbb{R} \,|\, n \geq 3\}\,.$$

In this paper, we frequently utilize sets that include all integers except for specific values, using the notations described above. Whenever such a set is introduced, its domain is briefly explained.

10.2 Double factorial

The double factorial [8] is not to be confused with the factorial function iterated twice (sequence A000197 in the OEIS), which is written as $(n!)!$ and not $n!!$. The discrepancy is shown in these examples:

$$(3!)! = 6! = 720\,,$$
$$3!! = 3 \cdot 1 = 3\,.$$

The term *odd factorial* is sometimes used to refer to the double factorial of an odd number, while *semifactorial* has been used by Knuth as a synonym for double factorial. We will not use these terms here.

In a 1902 paper, the physicist Arthur Schuster wrote about the double factorial:

> I propose to write $n!!$ for such products, and if a name be required for the product to call it the "alternate factorial" or the "double factorial".

Bruce Elwyn Meserve states in 1948 that the double factorial was originally introduced in order to simplify the expression of certain trigonometric integrals that arise in the derivation of the Wallis product. Double factorials also arise in expressing the volume of a hypersphere, and they have many applications in enumerative combinatorics.

An interesting property of the traditional factorial is that it can be expressed as the product of two double factorials. This can be illustrated with the following examples:

$$\begin{aligned} 7! &= 7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1 & 6! &= 6\cdot5\cdot4\cdot3\cdot2\cdot1 \\ &= 7\cdot5\cdot3\cdot1\cdot6\cdot4\cdot2 & &= 6\cdot4\cdot2\cdot5\cdot3\cdot1 \\ &= 7!!\cdot6!!\,, & &= 6!!\cdot5!!\,. \end{aligned}$$

This property is formalized in the following equations:

$$n! = n!! \cdot (n-1)!!\,, \quad n \in \mathbb{Z}^+, \qquad \text{(Eq. 10.2.1)}$$

$$n!! = \frac{(n+1)!}{(n+1)!!}\,, \quad n \in \mathbb{Z}_0^+. \qquad \text{(Eq. 10.2.2)}$$

We will list two more double factorial identities. Firsly, for an even non-negative integer $n = 2k$ with $k \in \mathbb{Z}_0^+$, the double factorial may be expressed as

$$(2k)!! = 2^k k!\,, \quad k \in \mathbb{Z}_0^+. \qquad \text{(Eq. 10.2.3)}$$

Secondly, for odd $n = 2k-1$ with $k \in \mathbb{Z}^+$, combining Eq. 10.2.2 and Eq. 10.2.3 yields

$$(2k-1)!! = \frac{(2k)!}{2^k k!}\,, \quad k \in \mathbb{Z}_0^+. \qquad \text{(Eq. 10.2.4)}$$

Furthermore, Ramanujan found this closed-form sum that relates $n!!$ with the Gamma function:

$$\sum_{n=0}^{\infty} (-1)^n \left[\frac{(2n-1)!!}{(2n)!!}\right]^3 = \left[\frac{\Gamma(^9/_8)}{\Gamma(^5/_4)\Gamma(^7/_8)}\right]^2 \qquad \text{(Eq. 10.2.5)}$$

10.3 Gamma function

The Gamma function is fundamental in many mathematical fields, including analysis, combinatorics, and number theory. It also has widespread applications in probability theory, particularly in continuous probability distributions.

Mathematically, it is expressed by the following integral:

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx \,, \quad x \in \mathbb{C} \setminus \mathbb{Z}^- . \quad \text{(Eq. 10.3.1)}$$

The Gamma function is defined for all complex numbers except non-positive integers. It is visualized graphically for real numbers through this plot:

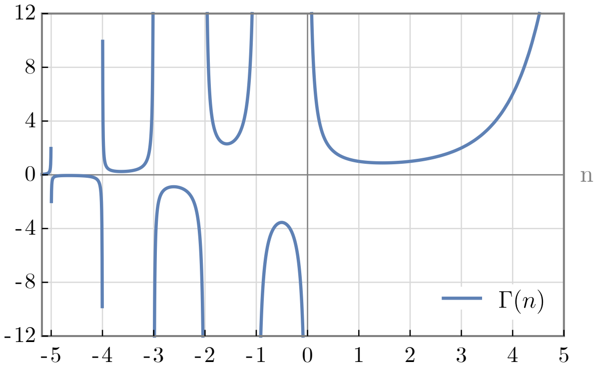

Fig. 10.3.1: The Gamma function $\Gamma(z)$

It is closely related to the traditional factorial by the equation written below[16]. There is an offset of $+1$, which exists only because of how the Gamma function is defined in Eq. 10.3.1:

$$\Gamma(n+1) = n! \,, \quad n \in \mathbb{R} \setminus \mathbb{Z}^- . \quad \text{(Eq. 10.3.2)}$$

Notably, the factorial as defined by the Gamma function is recursive, following the relationship:

$$n! = n(n-1)! \,, \quad n \in \mathbb{R} \setminus \mathbb{Z}^- . \quad \text{(Eq. 10.3.3)}$$

The recursive property of the factorial can be rewritten using the Gamma function, like so:

$$\Gamma(n) = n \cdot \Gamma(n-1) \,, \quad n \in \mathbb{R} \setminus \mathbb{Z}^- . \quad \text{(Eq. 10.3.4)}$$

Therefore, the Gamma function allows us to represent the factorial for all real numbers except negative integers. It takes the form:

$$n! = \int_0^\infty x^n e^{-x} dx \,, \quad n \in \mathbb{R} \setminus \mathbb{Z}^- . \quad \text{(Eq. 10.3.5)}$$

Alternatively, it can be written as:

$$n! = \int_0^\infty e^{-\sqrt[n]{x}} dx \,, \quad n \in \mathbb{R} \setminus \mathbb{Z}^- . \quad \text{(Eq. 10.3.6)}$$

[16] In the last part of this study, we will propose a factorial expansion to the domain of complex numbers, alternative and independent to the Gamma function.

Let's briefly discuss an important property of the Gamma function: it is logarithmically convex. This means that $ln\big(\Gamma(n)\big)$ is a convex function, as per the Bohr-Mollerup theorem.

Graphically, this means that for large enough z, the curve in Fig. 10.3.1 is "bowed" upward in a consistent manner. In other words, the slope of the curve increases in a smooth way as it moves from left to right along the x-axis.

Logarithmic convexity is a property that has important implications in various mathematical contexts, including optimization problems, probability theory, and statistical analysis. The Gamma function is the only continuation of the factorial to non-integers that has this property.

The double factorial has a continuous definition, related to the Gamma function [4]. It is iterated as follows:

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n-1)!!}{2^n} \sqrt{\pi} \,, \quad \text{(Eq. 10.3.7)}$$

$$\Rightarrow n!! = \sqrt{\frac{2}{\pi}} \, 2^{\frac{n}{2}} \, \Gamma\left(\frac{n}{2} + 1\right) . \quad \text{(Eq. 10.3.8)}$$

Interestingly, this formula yields different results for even integers. Specifically, they are all multiplied by $\sqrt{2/\pi}$, a fraction that does not simplify with the Gamma function at these values. Odd integers are calculated as expected, with remedies for this discrepancy explored shortly.

Let's now analyze what we used for Fig. 2.1.1 and Fig. 5.1.1, which are the figures that depict continuous extensions of the multifactorial. For the traditional factorial ($m = 1$), we used Eq. 10.3.2:

$$n!_{(1)} = \Gamma(n+1) \,, \quad n \in \mathbb{R}_0^+ . \quad \text{(Eq. 10.3.2)}$$

For $m = 2$, we applied a variation of Eq. 10.3.8, as it is defined in Mathematica [9]:

$$n!_{(2)} = \left(\frac{2}{\pi}\right)^{\frac{1-\cos n\pi}{4}} 2^{\frac{n}{2}} \, \Gamma\left(\frac{n}{2} + 1\right) , \quad n \in \mathbb{R}_0^+ . \quad \text{(Eq. 10.3.9)}$$

The trigonometric exponent ensures that all even double factorials are smoothly adjusted to remove the $\sqrt{2/\pi}$ term without altering odd values.

For $m = 3$ and $m = 4$, we employed the following $\prod$-product, discussed in a forum about triple factorial continuations [10]:

$$n!_{(m)} = m^{\frac{n}{m}} \, \Gamma\left(1 + \frac{n}{m}\right) \prod_{k=1}^{m} \left(\frac{k\, m^{-\frac{k}{m}}}{\Gamma(\frac{k}{m} + 1)}\right)^{f(n,m,k)} , \quad \text{(Eq. 10.3.10)}$$

where

$$f(n,m,k) = 1 + \frac{2\cos\frac{2\pi(n-k)}{m}}{m} . \quad \text{(Eq. 10.3.11)}$$

These relationships are used solely for generating Fig. 2.1.1 and Fig. 5.1.1, as non-integer factorials will be introduced later in this study.

10.4 $\prod$-product

The product operator $\prod$, denoted by the capital Greek letter "pi" (Π), is used to represent the product of a sequence, analogous to how $\sum$ represents summation [11]. For example, the product of the first 6 squares of natural numbers can be written as:

$$\prod_{k=1}^{6} k^2 = 1 \cdot 4 \cdot 9 \cdot 16 \cdot 25 \cdot 36\,.$$

The number above $\prod$ is the upper limit, and the one below is the lower limit. In the above example, the upper limit is 6, and the lower limit is 1. The variable k represents the multiplicands or factors.

If the product terms increase successively, it's termed a rising product, while if they decrease, it's known as a falling product.

Specifically, in this paper, we use the terms rising and falling products to refer to the multiplication of terms that are negative reciprocals of integers. This distinction relies on the expression involving k: if the product includes an expression like $n \pm k$, it is considered a falling product. Conversely, if n is absent, the product is regarded as a rising product.

For the $\prod$-product to be well-defined, the upper and lower limits are typically integers, usually natural numbers. When both limits are set to a specific number, the product evaluates to that number:

$$\prod_{k=5}^{5} k = 5\,, \quad \prod_{k=3}^{3} e^k = e^3\,.$$

If there are no factors at all, it results in an empty product, defined as 1. This happens when the upper limit is less than the lower limit, regardless of k:

$$\prod_{k=3}^{2} k = \prod_{k=3}^{2} 2k = \prod_{k=3}^{2} k^3 = 1\,.$$

In summary, the identities are:

$$\prod_{k=n}^{n} f(k) = f(n)\,, \quad \prod_{k=n}^{n-a} f(k) = 1\,, \quad a \in \mathbb{Z}^+, \quad n \in \mathbb{Z}\,, \qquad \text{(Eq. 10.4.1)}$$

where $f(k)$ is any function of k.

10.5 Floor, ceiling and sawtooth functions

The floor, ceiling, and sawtooth functions are fundamental mathematical tools, particularly useful for operations involving rounding and modular arithmetic. In this subsection, we introduce these functions and explore some of their key properties.

We begin with the floor function, which rounds a number down to the nearest integer. For example:

$$\lfloor 1.3 \rfloor = 1\,, \qquad \lfloor 2.7 \rfloor = 2\,,$$

$$\lfloor 3 \rfloor = 3\,, \qquad \lfloor -3.4 \rfloor = -4\,.$$

The graph of the $\lfloor n \rfloor$ is shown below:

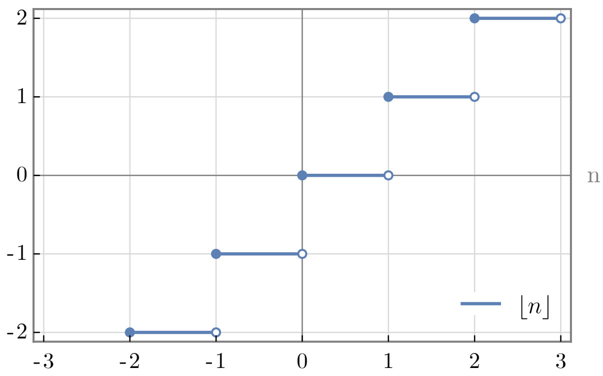

Fig. 10.5.1: The floor function $\lfloor n \rfloor$

This graph resembles a set of steps, which is why the floor function is sometimes referred to as a "step function." However, it is not technically a step function because it involves an infinite number of intervals, or "steps."

The formal definition of the floor function is given by:

$$\lfloor n \rfloor = max\{m \in \mathbb{Z} \,|\, m \leq n\}\,. \qquad \text{(Eq. 10.5.1)}$$

Next, we present the ceiling function, which effectively rounds a number up to the nearest integer. This is illustrated as follows:

$$\lceil 0.8 \rceil = 1\,, \qquad \lceil 1.2 \rceil = 2\,,$$

$$\lceil 3 \rceil = 3\,, \qquad \lceil -4.6 \rceil = -4\,.$$

The plot of $\lceil n \rceil$ is displayed below:

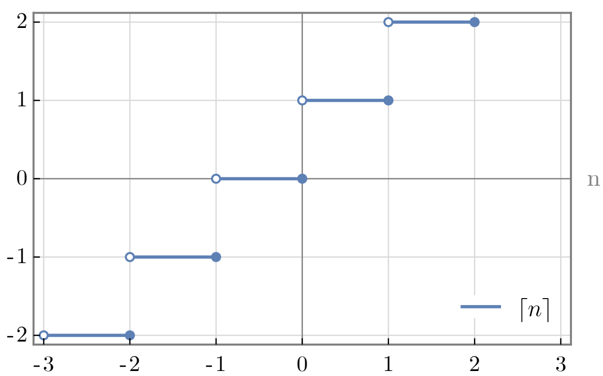

Fig. 10.5.2: The ceiling function $\lceil n \rceil$

This plot differs slightly from that of the floor function. Notably, it is shifted one unit to the left or one unit upward. The filled-in dots on the graph align with the integers on the line $f(n) = n$, as both the floor and ceiling functions return integer values at these instances.

The ceiling function is formally defined as:

$$\lceil n \rceil = min\{m \in \mathbb{Z} \,|\, m \geq n\}\,. \qquad \text{(Eq. 10.5.2)}$$

Lastly, we introduce the sawtooth function, also known as the fractional part of a number. It is defined as:

$$\{n\} = n - \lfloor n \rfloor\,, \quad n \in \mathbb{R}\,. \qquad \text{(Eq. 10.5.3)}$$

This definition is intuitive since every number has an integer and a fractional part, which are summed together to make up that number. Therefore, the sawtooth function is the difference between n and $\lfloor n \rfloor$, a number that is always between 0 and 1. The following examples help to grasp this concept:

$$\begin{aligned} \{5.7\} &= 5.7 - \lfloor 5.7 \rfloor = 5.7 - 5 = 0.7\,, \\ \{6\} &= 6 - \lfloor 6 \rfloor = 6 - 6 = 0\,, \\ \{-0.2\} &= -0.2 - \lfloor -0.2 \rfloor = -0.2 - (-1) = 0.8\,, \\ \{-3.9\} &= -3.9 - \lfloor -3.9 \rfloor = -3.9 - (-4) = 0.1\,. \end{aligned}$$

The sawtooth function is named like that because its graph resembles the sharp, jagged edge of a saw:

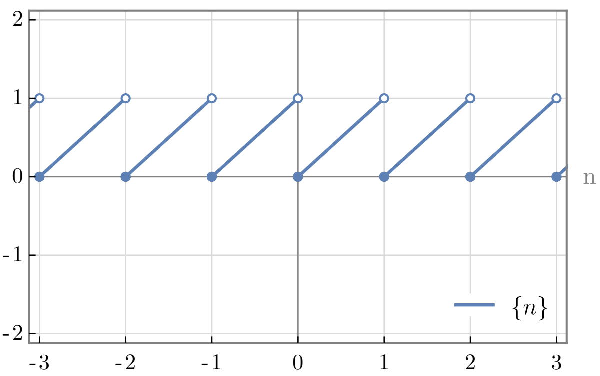

Fig. 10.5.3: The sawtooth function $\{n\}$

With a basic understanding of these functions, we can now discuss a few key properties [12]. For any real number x and integer n, the following identities hold:

$$\left.\begin{aligned} \lfloor x+n \rfloor &= \lfloor x \rfloor + n \\ \lceil x+n \rceil &= \lceil x \rceil + n \\ \{x+n\} &= \{x\} \end{aligned}\right\} x \in \mathbb{R}\,, \quad n \in \mathbb{Z}\,. \qquad \begin{aligned} &\text{(Eq. 10.5.4)} \\ &\text{(Eq. 10.5.5)} \\ &\text{(Eq. 10.5.6)} \end{aligned}$$

The first two equations indicate that adding an integer to a number does not affect the result when applying the floor or ceiling functions, since the integer can be factored out. The third equation shows that the fractional part of a number remains unchanged by adding an integer to it.

There is also a connection to modular arithmetic:

$$x \bmod y = x - y\left\lfloor \frac{x}{y} \right\rfloor\,, \quad x \in \mathbb{R}\,, \quad y \in \mathbb{R}^*. \qquad \text{(Eq. 10.5.7)}$$

Additionally, for $x, y \in \mathbb{R}$, we have:

$$\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x+y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1\,, \quad x, y \in \mathbb{R}\,, \qquad \text{(Eq. 10.5.8)}$$

$$\lceil x \rceil + \lceil y \rceil - 1 \leq \lceil x+y \rceil \leq \lceil x \rceil + \lceil y \rceil\,, \quad x, y \in \mathbb{R}\,. \qquad \text{(Eq. 10.5.9)}$$

These properties constrain the sum of the floor or ceiling of two numbers within the sum of their individual floors or ceilings, plus or minus one. Although these specific properties are not used in this paper, they are included for future reference.

Moreover, the floor, ceiling, and sawtooth functions are idempotent[17]:

$$\left.\begin{aligned} \lfloor \lfloor x \rfloor \rfloor &= \lfloor x \rfloor \\ \lceil \lceil x \rceil \rceil &= \lceil x \rceil \\ \{\{x\}\} &= \{x\} \end{aligned}\right\} x \in \mathbb{R}\,. \qquad \begin{aligned} &\text{(Eq. 10.5.10)} \\ &\text{(Eq. 10.5.11)} \\ &\text{(Eq. 10.5.12)} \end{aligned}$$

Interestingly, negating the argument of these functions switches the floor and ceiling while changing the sign:

$$\lfloor x \rfloor = -\lceil -x \rceil\,, \quad x \in \mathbb{R}\,. \qquad \text{(Eq. 10.5.13)}$$

This property can be visualized by reflecting the graph of the ceiling function (as shown in Fig. 10.5.2) on its vertical axis and then on its horizontal axis. This geometric perspective helps to intuitively understand the relationship between the floor and ceiling functions.

For the last set of properties, we will present a modified *foundational function* from Part 1. It is defined as follows:

$$Q\{n\} \equiv Q\big(\{n\}\big) = \begin{cases} 0\,, & n \in \mathbb{Z} \\ 1\,, & n \notin \mathbb{Z}\,. \end{cases} \qquad \text{(Eq. 10.5.14)}$$

Coincidentally, this function is equivalent to the expression $\lceil \{n\} \rceil$. Using it, we can express the following identities:

$$\left.\begin{aligned} \lfloor x \rfloor + \lfloor -x \rfloor &= -Q\{x\} \\ \lceil x \rceil + \lceil -x \rceil &= Q\{x\} \\ \{x\} + \{-x\} &= Q\{x\} \end{aligned}\right\} x \in \mathbb{R}\,. \qquad \begin{aligned} &\text{(Eq. 10.5.15)} \\ &\text{(Eq. 10.5.16)} \\ &\text{(Eq. 10.5.17)} \end{aligned}$$

This concludes our discussion of the floor, ceiling, and sawtooth functions. As all *F.F.* are derived from these basic functions, they serve as the foundational building blocks of this study, making a rigorous exploration of their properties essential.

10.6 Modular arithmetic

Modular arithmetic [13] is a system of arithmetic where numbers "wrap around" upon reaching a specific value, known as the modulus. This concept was formalized by Carl Friedrich Gauss in his book *Disquisitiones Arithmeticae*, published in 1801.

A common example of modular arithmetic is the 12-hour clock. In this system, time is divided into two 12-hour periods. For instance, if it is currently 7:00, then 8 hours later it will be 3:00. While

[17] Idempotence was briefly mentioned in Subsection 5.4. In this context, taking the floor of $\lfloor x \rfloor$ is equal to $\lfloor x \rfloor$, as iterated in Eq. 10.5.10.

straightforward addition yields $7+8=15$, the clock "wraps around" every 12 hours, so 15:00 is interpreted as 3:00. This relationship can be expressed as $15 \equiv 3 \bmod 12$, which means that $7+8 \equiv 3 \bmod 12$.

Similarly, an 8-hour period can be represented as 8:00, and doubling this period gives 16:00, which is 4:00 on the clock face. This can be written as $2 \times 8 \equiv 4 \bmod 12$. The symbol "≡" denotes equivalence in modular arithmetic, which indicates that two quantities are congruent modulo a given value rather than being strictly equal in the traditional sense.

Consider the following examples:

$$\begin{gathered}\text{In mod 10: } 36 \equiv 26 \equiv 16 \equiv 6\,, \\ \text{In mod 12: } 50 \equiv 38 \equiv 26 \equiv 14 \equiv 2\,, \\ \text{In mod 2: } 14 \equiv 30 \equiv 2 \equiv 0\,,\quad 11 \equiv 51 \equiv 3 \equiv 1\,, \\ \Rightarrow \text{In mod 2: even } n \equiv 0\,,\quad \text{odd } n \equiv 1\,.\end{gathered}$$

In these examples, we observe that modular operations often yield results of 0. Generally, for any integer n and positive integer m, if n is divisible by m, the result of the modular operation is zero:

$$n \bmod m = 0\,,\quad m,\ \frac{n}{m} \in \mathbb{Z}^+,\quad n \in \mathbb{Z}_0^+\,. \tag{Eq. 10.6.1}$$

Modular arithmetic can also be extended to negative values, which is necessary for the purposes of this paper. Here are a few examples:

$$\begin{gathered}\text{In mod 5: } 15 \equiv 10 \equiv 5 \equiv 0 \equiv -5 \equiv -10\,, \\ \text{In mod 2: } 1 \equiv -1 \equiv -3\,,\quad 4 \equiv 0 \equiv -6 \equiv -18\,, \\ \text{In mod 10: } 16 \equiv 6 \equiv -4 \equiv -14 \equiv -24\,, \\ \text{In mod 12: } 14 \equiv 2 \equiv -10 \equiv -22 \equiv -34\,.\end{gathered}$$

To conclude, we will analyze a specific expression related to modular arithmetic that is introduced in Subsection 7.4. This expression is always an integer and is defined as:

$$f(n,m) = \frac{n - n \bmod m}{m} + 1\,,\quad n \in \mathbb{Z}\,. \tag{Eq. 10.6.2}$$

The function $f(n,m)$ is temporarily named for convenience in this discussion. Note that m is always a natural number in the context of this paper.

To better understand why $f(n,m)$ is an integer, we can simplify it by temporarily omitting the +1, as it does not affect the integer nature of the result.

For $m=1$, the expression simplifies to:

$$f(n,1) - 1 = \frac{n - n \bmod 1}{1} = n\,,\quad n \in \mathbb{Z}\,. \tag{Eq. 10.6.3}$$

Here, $n \bmod 1$ represents the fractional part of n. Since $n \bmod 1 = 0$ for integer n, this term simplifies and does not appear in the equation above.

For $m=2$, the expression becomes:

$$f(n,2) - 1 = \frac{n - n \bmod 2}{2} = \frac{n}{2} - \frac{n \bmod 2}{2}\,,\ n \in \mathbb{Z}\,. \tag{Eq. 10.6.4}$$

To determine whether $f(n,2) \in \mathbb{Z}$, we can examine two cases based on whether n is even or odd. For even n, let $n = 2k$, and for odd n, let $n = 2k+1$, where $k \in \mathbb{Z}$.

Substituting these into $f(n,2)$ results in:

$$\begin{cases} \dfrac{2k - (2k) \bmod 2}{2} = \dfrac{2k - 0}{2} = k \\[2ex] \dfrac{2k + 1 - (2k+1) \bmod 2}{2} = \dfrac{2k+1-1}{2} = k\,. \end{cases}$$

In both cases, the result is k, which is an integer. Specifically, $2k \bmod 2 = 0$ for even n, and for odd n, $(2k+1) \bmod 2 = 1$.

For the general case, where $n = mk + l$ and l is an integer between 0 and $m-1$, we have:

$$\frac{mk + l - (mk+l) \bmod m}{m} = \frac{mk + l - l}{m} = k\,.$$

Thus, $f(n,m)$ is always an integer for any integer n. This property is crucial because in Subsection 7.4, we define $\eta(n)$ as $(-1)^{f(n,m)}$ for negative integers, and this function is required to produce real numbers.

As a final consideration, let's explore the scenario where n is a real number rather than an integer. In this case, the expression $n = mk + l$ would result in the variable l being a real number, falling within the interval $[0,\, n)$. Upon performing the same calculations as in the previous example, we observe that the outcome remains unchanged. Consequently, the function $f(n,m)$ is well-defined for all $n \in \mathbb{R}$.